Practice Problems

1. The first three terms of an arithmetic sequence are 7, 9.5, 12.

(a) What is the 41st term of the sequence?

(b) What is the sum of the first 101 terms of the sequence?

(a) $d = 2.5$

$a_n = 7 + 2.5(n-1)$

$a_n = 2.5n + 4.5$

$a_{41} = 2.5(41) + 4.5$

$\boxed{a_{41} = 107}$

(b) $a_1 = 7$ $a_{101} = 2.5(101) + 4.5$

$a_{101} = 257$

$S_{101} = \frac{101}{2}(7 + 257)$

$\boxed{S_{101} = 13{,}332}$

(Total 4 marks)

2. In the arithmetic series with n^{th} term u_n, it is given that $u_4 = 7$ and $u_9 = 22$.
Find the minimum value of n so that $u_1 + u_2 + u_3 + \ldots + u_n > 10\ 000$.

$d = \frac{22-7}{9-4}$

$d = \frac{15}{5}$

$d = 3$

$S_n = \frac{n}{2}(a_1 + a_n)$

$a_n = a_1 + d(n-1)$

$a_1 + 3d = a_4$

$a_1 = 7 - 9$

$a_1 = -2$

$a_n = -2 + 3(n-1)$

$a_n = 3n - 5$

$S_n = \frac{n}{2}(-2 + 3n - 5)$

$10000 = \frac{n}{2}(3n - 7)$

$3n^2 - 7n - 20{,}000 = 0$

$n = \frac{7 \pm \sqrt{49 - 4(3)(-20000)}}{6}$

$n = 82.8, -80.49$

$\boxed{n = 83}$

(Total 5 marks)

3. Consider the infinite geometric sequence 3000, – 1800, 1080, – 648, … .

(a) Find the common ratio. **(2)**

(b) Find the 10th term. **(2)**

(c) Find the **exact** sum of the infinite sequence. **(2)**

(a) $r = \frac{-1800}{3000}$

$\boxed{r = -\frac{3}{5} \text{ or } -0.6}$

(b) $a_n = 3000 \cdot \left(-\frac{3}{5}\right)^{n-1}$

$a_{10} = 3000 \cdot \left(-\frac{3}{5}\right)^9$

$\boxed{a_{10} = -30.233}$

(c) $S = \frac{a_1}{1-r}$

$S = \frac{3000}{1-\left(-\frac{3}{5}\right)}$

$S = \frac{3000}{\frac{8}{5}}$

$\boxed{S = 1875}$

(Total 6 marks)

4. The sum of an infinite geometric sequence is $13\frac{1}{2}$, and the sum of the first three terms is 13. Find the first term.

$13\frac{1}{2} = \frac{a_1}{1-r}$ $\quad 13 = \frac{a_1(1-r^3)}{1-r}$

SOLVE FOR $\frac{a_1}{1-r}$

$\frac{a_1}{1-r} = 13\frac{1}{2}$ | $\frac{a_1}{1-r} = \frac{13}{1-r^3}$

$13\frac{1}{2} = \frac{13}{1-r^3}$

$1-r^3 = \frac{13}{13\frac{1}{2}}$

$1-r^3 = \frac{26}{27}$

$r^3 = \frac{1}{27}$

$r = \frac{1}{3}$

$\frac{a_1}{1-\frac{1}{3}} = 13\frac{1}{2}$

$\boxed{a_1 = 9}$

(Total 3 marks)

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5. Find the coefficient of x^3 in the binomial expansion of $\left(1-\frac{1}{2}x\right)^8$.

$\binom{8}{5}(1)^5\left(-\frac{1}{2}x\right)^3 \longrightarrow 56 \cdot 1 \cdot \left(-\frac{1}{8}x^3\right)$

$\binom{8}{5} = \binom{8}{3} = \frac{8!}{5!\,3!} = 56$

$\boxed{-7}x^3$

COEFFICIENT

(Total 4 marks)

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6. Find the constant term in the binomial expansion of $\left(2x^2 - \frac{1}{x}\right)^9$.

$\binom{9}{3}(2x^2)^3\left(-\frac{1}{x}\right)^6 = 84 \cdot 8x^6 \cdot \frac{1}{x^6}$

$= \boxed{672}$

(Total 6 marks)

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7. Find the sum of all three-digit natural numbers that are not exactly divisible by 3.

ALL (100 - 999)

$S_{900} = \frac{900}{2}(100+999)$

$S_{900} = 494,550$

DIV BY 3 (102 - 999)

$S_{300} = \frac{300}{2}(102+999)$

$S_{300} = 165,150$

NOT DIV. BY 3

$494550 - 165150$

$\boxed{329,400}$

(Total 5 marks)

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8. How many four-digit numbers are there which contain at least one digit 3?

ALL 4 DIGIT #'S $\quad 9 \cdot 10 \cdot 10 \cdot 10 = 9,000$

NO 3'S $\quad 8 \cdot 9 \cdot 9 \cdot 9 = 5,832$

CAN'T BEGIN W/ 0

$\boxed{3,168}$

(Total 3 marks)

9. A local bridge club has 17 members, 10 females and 7 males. They have to elect 3 officers: president, deputy, and treasurer. In how many ways is this possible if:

ORDER IMPORTANT → PERMUTATION

(a) there are no restrictions? **(2)**

$$17\cdot16\cdot15 = \boxed{4,080}$$

(b) the president is male? **(2)**

$$7\cdot16\cdot15 = \boxed{1,680}$$

(c) the president and deputy are the same gender? **(2)**

$$\underbrace{7\cdot6}_{\text{MALE}}\cdot\underbrace{15}_{\text{ANY}} + \underbrace{10\cdot9}_{\text{FEMALE}}\cdot\underbrace{15}_{\text{ANY}} = 630+1350 = \boxed{1980}$$

(Total 6 marks)

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10. Let f and g be two functions. Given that $(f\circ g)(x) = \dfrac{x+1}{2}$ and $g(x) = 2x - 1$, find $f(x - 3)$.

$$f(g(x)) = \frac{x+1}{2}$$

$$f(2x-1) = \frac{x+1}{2}$$

$g^{-1}(x)$

$y = 2x-1$

$x = 2y-1$

$x+1 = 2y$

$y = \dfrac{x+1}{2}$

$$(f\circ \cancel{g\circ g^{-1}})(x) = \frac{\left(\frac{x+1}{2}\right)+1}{2} = \frac{x+3}{4}$$

$$f(x) = \frac{x+3}{4}$$

$$f(x-3) = \frac{x-3+3}{4}$$

$$\boxed{f(x-3) = \frac{x}{4}}$$

(Total 6 marks)

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11. The function f is defined by $f : x \mapsto x^3$.
Find an expression for $g(x)$ in terms of x in each of the following cases

(a) $(f\circ g)(x) = x + 1$; $\quad g(x) = \sqrt[3]{x+1} \longrightarrow f(g(x)) = \left(\sqrt[3]{x+1}\right)^3 = x+1$

(b) $(g\circ f)(x) = x + 1$. $\quad g(x) = \sqrt[3]{x}+1 \longrightarrow g(f(x)) = \sqrt[3]{x^3+1} = x+1$

(Total 6 marks)

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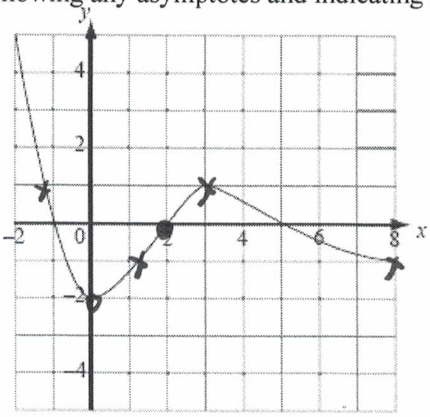
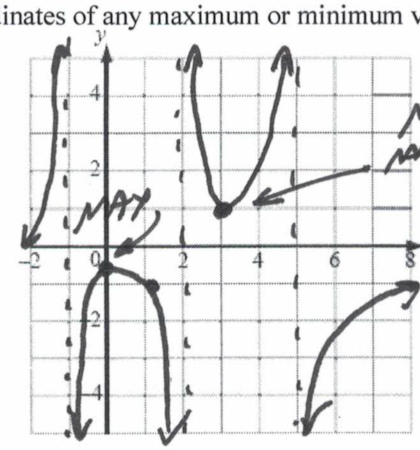
12. The graph of $y = f(x)$ for $-2 \le x \le 8$ is shown. On the set of axes provided, sketch the graph of $y = \dfrac{1}{f(x)}$, clearly showing any asymptotes and indicating the coordinates of any maximum or minimum values.





MIN
~~MAX~~ → (3, 1)

(Total 5 marks)

13. The functions f and g are defined below. Find the values of x for which $(f \circ g)(x) \le (g \circ f)(x)$.

$$f(x) = 2x - 1 \qquad g(x) = \frac{x}{x+1}, x \ne -1.$$

SOLUTION

$(-1, 0) \cup [\frac{1}{3}, \infty)$

or

$-1 < x < 0$ AND $x \ge \frac{1}{3}$

$f(g(x)) = 2\left(\frac{x}{x+1}\right) - 1 = \frac{2x}{x+1} - \frac{x+1}{x+1} = \boxed{\frac{x-1}{x+1}}$

$g(f(x)) = \frac{2x-1}{(2x-1)+1} = \boxed{\frac{2x-1}{2x}}$

$\frac{x-1}{x+1} \le \frac{2x-1}{2x}$ CRITICAL PTS $x = -1, 0$ (DENOM $= 0$)

$\frac{x-1}{x+1} \times \frac{2x-1}{2x}$

$2x(x-1) = (2x-1)(x+1)$

$2x^2 - 2x = 2x^2 + 2x - x + 1$

$-3x = -1$

$x = \frac{1}{3}$

TEST VALUES $\left(\frac{x+1}{x-1} \le \frac{2x-1}{2x}\right)$

✗ ✓ ✗ ✓

-1 0 $1/3$

$3 \le \frac{5}{4}$ NO! $-3 \le 1$ YES! $-\frac{3}{5} \le -1$ NO! $0 \le \frac{1}{2}$ YES

(Total 6 marks)

14. Let $f(x) = \frac{4}{x+2}, x \ne -2$ and $g(x) = x - 1$.

If $h = g \circ f$, find

(a) $h(x)$; $g(f(x)) = \frac{4}{x+2} - 1 = \frac{4-(x+2)}{x+2} = \frac{2-x}{x+2}$ **(2)**

(b) $h^{-1}(x)$, where h^{-1} is the inverse of h. **(4)**

$y = \frac{4}{x+2} - 1$

$x = \frac{4}{y+2} - 1$

$x + 1 = \frac{4}{y+2}$

$y + 2 = \frac{4}{x+1}$

$y = \frac{4}{x+1} - 2$

$y = \frac{4 - 2(x+1)}{x+1}$

$\boxed{y = \frac{2-2x}{x+1}}$

(Total 6 marks)

15. Given functions $f(x) = 2x + 1$ and $g(x) = x^3$, find the function $(f^{-1} \circ g)^{-1}$.

$y = 2x + 1$

$x = 2y + 1$

$2y = x - 1$

$y = \frac{x-1}{2}$

$f^{-1}(g(x)) = \frac{x^3 - 1}{2}$

$y = \frac{x^3 - 1}{2}$

$x = \frac{y^3 - 1}{2}$

$2x = y^3 - 1$

$y^3 = 2x + 1$

$\boxed{y = \sqrt[3]{2x+1}}$

(Total 4 marks)

16. The function f is defined for $x \le 0$ by $f(x) = \dfrac{x^2-1}{x^2+1}$. Find an expression for $f^{-1}(x)$.

$y = \dfrac{x^2-1}{x^2+1}$

$x = \dfrac{y^2-1}{y^2+1}$

$x(y^2+1) = y^2-1$

$xy^2 + x = y^2 - 1$

$xy^2 - y^2 = -x - 1$

$y^2(x-1) = -x-1$

$y^2 = \dfrac{-x-1}{x-1}$

$y = \pm\sqrt{\dfrac{x+1}{1-y}}$

$\boxed{y = -\sqrt{\dfrac{x+1}{1-y}}}$

IF $x \le 0$, THEN $y \le 0$ FOR f^{-1}

(Total 6 marks)

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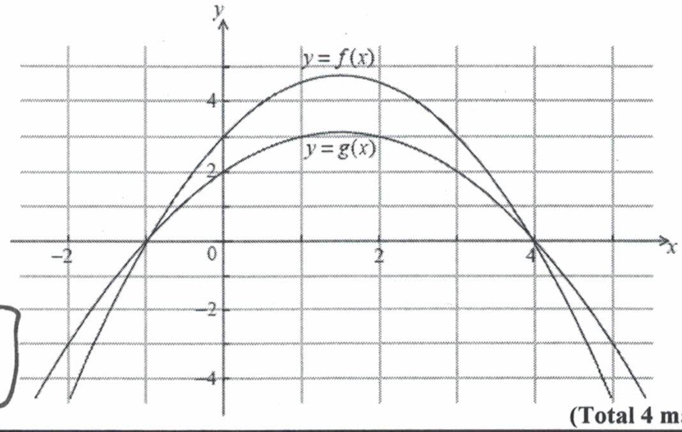
17. Shown below are the graphs of $y = f(x)$ and $y = g(x)$.
If $(f \circ g)(x) = 3$, find all possible values of x.



$f(g(x)) = 3$

$f(x) = 3$ @ 0 AND 3

$g(x) = 0$ @ $-1, 4$

$g(x) = 3$ @ $1, 2$

$\boxed{x = -1, 1, 2, 4}$

(Total 4 marks)

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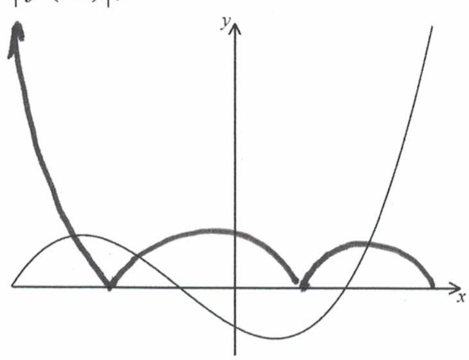
18. Each of the diagrams below shows the graph of a function f. Sketch on the given axes the graph of

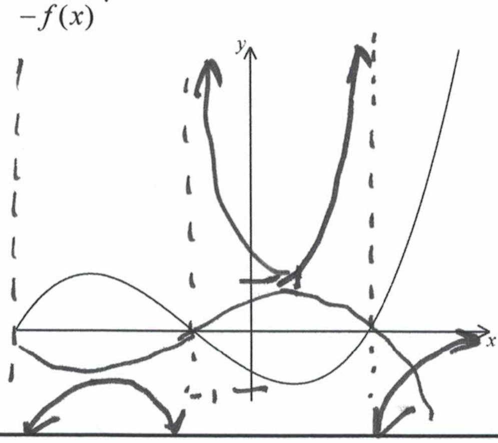
(a) $|f(-x)|$;



$f(-x) \rightarrow$ FLIPS OVER Y-AXIS

$|f(-x)| \rightarrow$ BELOW X-AXIS FLIPS UP.

(b) $\dfrac{1}{-f(x)}$.



$-f(x) \rightarrow$ FLIPS OVER X-AXIS

$\dfrac{1}{-f(x)} \rightarrow$ BIG → SMALL
SMALL → BIG
ZERO → ASYMP.
$y = \pm 1$ → STAYS SAME

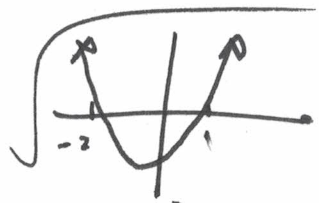
(Total 6 marks)

19. The functions $f(x)$ and $g(x)$ are given by $f(x) = \sqrt{x-2}$ and $g(x) = x^2 + x$. The function $(f \circ g)(x)$ is defined for $x \in \mathbb{R}$, **except** for the interval $]\, a, b\, [$.

(a) Calculate the value of a and of b.

(b) Find the range of $f \circ g$.

$f(g(x)) = \sqrt{x^2+x-2} = \sqrt{(x+2)(x-1)}$



(a) SEE GRAPH

$(x+2)(x-1) = 0$

$x = -2, 1$

$\sqrt{+}$, $\sqrt{-}$, $\sqrt{+}$

$\checkmark$ -2 $\times$ 1

$\boxed{a = -2 \quad b = 1}$

b.) $\boxed{\text{RANGE: } y \geq 0}$

(Total 6 marks)

20. Solve the equation $\log_3(x + 17) - 2 = \log_3 2x$.

$\log_3 (x+17) - 2\log_3 3 = \log_3 2x$

$\log_3 \left(\frac{x+17}{3^2}\right) = \log_3 2x$

$\frac{x+17}{9} = 2x$

$x + 17 = 18x$

$\boxed{x = 1}$

(Total 5 marks)

21. Solve the equation $2^{2x+2} - 10 \times 2^x + 4 = 0, x \in \mathbb{R}$.

$2^{2x} \cdot 2^2 - 10 \cdot 2^x + 4 = 0$

$4(2^x)^2 - 10(2^x) + 4 = 0$

$y = 2^x \leftarrow$ MAKES IT EASIER, BUT NOT NECESSARY

$(4y^2 - 10y + 4 = 0) \cdot \frac{1}{2}$

$2y^2 - 5y + 2 = 0$

$(2y-1)(y-2) = 0$

$y = \frac{1}{2}, 2$

$2^x = \frac{1}{2} \quad 2^x = 2$

$\boxed{x = -1, 1}$

(Total 6 marks)

22. Find the **exact** value of x satisfying the equation

$$(3^x)(4^{2x+1}) = 6^{x+2}.$$

Give your answer in the form $\dfrac{\ln a}{\ln b}$ where $a, b \in \mathbb{Z}$.

$(3^x)(4^{2x+1}) = 6^{x+2}$

$(3^x)((2^2)^{2x+1}) = (2\cdot 3)^{x+2}$

$3^x \cdot 2^{4x+2} = 2^{x+2}\cdot 3^{x+2}$

$\dfrac{2^{4x+2}}{2^{x+2}} = \dfrac{3^{x+2}}{3^x}$

$2^{3x} = 3^2$

$2^{3x} = 9$

$\ln 2^{3x} = \ln 9$

$3x\cdot \ln 2 = \ln 9$

$x = \dfrac{\ln 9}{3\ln 2}$

$\boxed{x = \dfrac{\ln 9}{\ln 8}}$

(Total 6 marks)

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23. Solve the equation $9 \log_5 x = 25 \log_x 5$, expressing your answers in the form $5^{\frac{p}{q}}$, where $p, q \in \mathbb{Z}$.

CHANGE OF BASE

$9\cdot\dfrac{\log x}{\log 5} = 25\cdot\dfrac{\log 5}{\log x}$

$9(\log x)^2 = 25\cdot(\log 5)^2$

$\left(\dfrac{\log x}{\log 5}\right)^2 = \dfrac{25}{9}$

IGNORE $\pm$

$\dfrac{\log x}{\log 5} = \dfrac{5}{3}$

$\log_5 x = \dfrac{5}{3}$ LOG TO EXP. FORM

$x = 5^{5/3}$

(Total 5 marks)

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24. The solution of $2^{2x+3} = 2^{x+1} + 3$ can be expressed in the form $a + \log_2 b$ where $a, b \in \mathbb{Z}$. Find the value of a and of b.

$2^{2x}\cdot 2^3 = 2^x\cdot 2^1 + 3$

$8\cdot(2^x)^2 = 2\cdot(2^x) + 3$

$y = 2^x$

$8y^2 = 2y + 3$

$8y^2 - 2y - 3 = 0$

$(4y-3)(2y+1) = 0$

$y = \frac{3}{4}, -\frac{1}{2}$

$2^x = \frac{3}{4}$

$2^x = -\frac{1}{2}$ No SOLUTION

$x = \log_2 \frac{3}{4}$

$x = \log_2 3 - \log_2 4$

$\boxed{x = \log_2 3 - 2}$

on

$x = -2 + \log_2 3$

$\boxed{a = -2,\ b = 3}$

(Total 6 marks)

25. Solve, for x, the equation $\log_2 (5x^2 - x - 2) = 2 + 2 \log_2 x$.

$\log_2 (5x^2 - x - 2) = 2\log_2 2 + \log_2 x^2$

$\log_2 (5x^2 - x - 2) = \log_2 (4x^2)$

$5x^2 - x - 2 = 4x^2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, \cancel{-1}$ CANNOT $\log(-1)$

$\boxed{x = 2}$

(Total 5 marks)

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26. Write $\ln (x^2 - 1) - 2 \ln(x + 1) + \ln(x^2 + x)$ as a single logarithm, in its simplest form.

$\ln\left(\frac{(x+1)(x-1)(x)(x+1)}{(x+1)(x+1)}\right)$

$\ln (x(x-1))$

$\boxed{\ln (x^2 - x)}$

(Total 5 marks)

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27. Solve the equations

$$\ln \frac{x}{y} = 1$$

$$\ln x^3 + \ln y^2 = 5.$$

$(\ln x - \ln y = 1) \cdot 2$

$3\ln x + 2\ln y = 5$

$2\ln x - \cancel{2\ln y} = 2$

$3\ln x + \cancel{2\ln y} = 5$

$5\ln x = 7$

$\ln x = \frac{7}{5}$

$\frac{7}{5} - \ln y = 1$

$\ln y = \frac{2}{5}$

$\ln x = \frac{7}{5}$ $\quad$ $\ln y = \frac{2}{5}$

$\boxed{x = e^{7/5} \qquad y = e^{2/5}}$

(Total 5 marks)

28. Solve $2(5^{x+1}) = 1 + \frac{3}{5^x}$, giving the answer in the form $a + \log_5 b$, where $a, b \in \mathbb{Z}$.

$2\cdot 5^x \cdot 5^1 = 1 + \frac{3}{5^x}$

$10\cdot 5^x - 1 - \frac{3}{5^x} = 0$

$10(5^x)^2 - 5^x - 3 = 0$

$y = 5^x$

$10y^2 - y - 3 = 0$

$(5y-3)(2y+1) = 0$

$\underline{y = \frac{3}{5}}$ $\quad y = -\frac{1}{2}$

NO SOLUTION

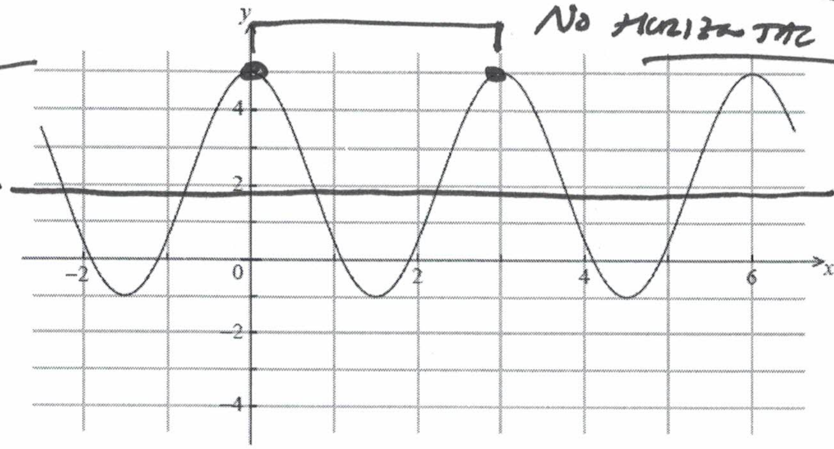
$5^x = \frac{3}{5}$

$\log_5 5^x = \log_5 \frac{3}{5}$

$\boxed{x = -1 + \log_5 3 \quad a = -1 \quad b = 3}$

(Total 6 marks)

29. The graph below shows $y = a\cos(bx) + c$.



NO HORIZONTAL SHIFT

$a = 3$

$c = +2$

PERIOD $= \frac{2\pi}{3}$

$3 = \frac{2\pi}{b}$

$b = \frac{2\pi}{3}$

Find the value of a, the value of b and the value of c.

$a = 3 \quad b = \frac{2\pi}{3} \quad c = 2$

(Total 4 marks)

30. The depth, $h(t)$ meters, of water at the entrance to a harbor at t hours after midnight on a particular day is given by:

$$h(t) = 8 + 4\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 24.$$

(a) Find the maximum depth and the minimum depth of the water. **(3)**

VERT. SHIFT $= +8$ (UP)

AMPLITUDE $= 4$

MAX $= 8 + 4 = 12$ m

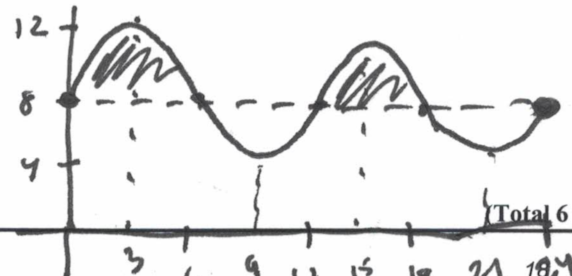
MIN $= 8 - 4 = 4$ m

(b) Find the values of t for which $h(t) \ge 8$. **(3)**

$8 + 4\sin\left(\frac{\pi}{6}t\right) \ge 8$

$\sin\left(\frac{\pi}{6}\cdot t\right) \ge 0$

$\boxed{0 \le t \le 6,\ 12 \le t \le 18,\ t = 24}$



(Total 6 marks)

31. Solve $\sin 2x = \sqrt{2}\cos x$, $0 \le x \le \pi$.

$2\sin x\cos x - \sqrt{2}\cos x = 0$

$\cos x(2\sin x - \sqrt{2}) = 0$

$\cos x = 0 \quad \sin x = \frac{\sqrt{2}}{2}$

$\cos x = 0 \qquad \sin x = \frac{\sqrt{2}}{2}$

$x = \frac{\pi}{2} \qquad x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}}$

(Total 6 marks)

32. The angle θ satisfies the equation $2\tan^2\theta - 5\sec\theta - 10 = 0$, where θ is in the second quadrant. Find the value of $\sec\theta$.

$\rightarrow \sec = -$

PYTH. IDENTITY: $1 + \tan^2\theta = \sec^2\theta$

$2(\sec^2\theta - 1) - 5\sec^2\theta - 10 = 0$

$2\sec^2\theta - 5\sec\theta - 12 = 0$

$(2\sec\theta + 3)(\sec\theta - 4) = 0$

$\sec\theta = -\frac{3}{2}, 4$

$\boxed{\sec\theta = -\frac{3}{2}}$

(Total 6 marks)

33. Given that $\tan 2\theta = \frac{3}{4}$, find the possible values of $\tan\theta$.

DOUBLE ∠ → SEE RACKET

$\frac{2\tan\theta}{1-\tan^2\theta} \times \frac{3}{4}$

$8\tan\theta = 3 - 3\tan^2\theta$

$3\tan^2\theta + 8\tan\theta - 3 = 0$

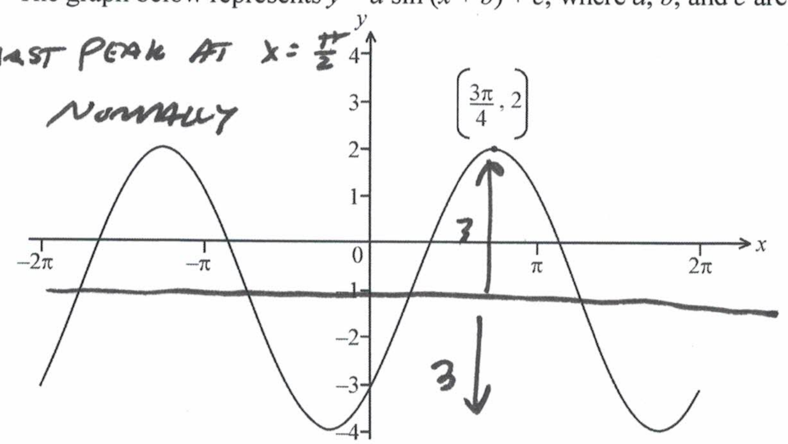
$(3\tan\theta - 1)(\tan\theta + 3) = 0$

$\boxed{\tan\theta = \frac{1}{3}, -3}$

(Total 5 marks)

34. The graph below represents $y = a\sin(x + b) + c$, where a, b, and c are constants.

FIRST PEAK AT $x = \frac{\pi}{2}$ NORMALLY



PERIOD → UNCHANGED

$\boxed{a = 3 \quad b = -\frac{\pi}{4} \quad c = -1}$

$y = 3\sin\left(x + \left(-\frac{\pi}{4}\right)\right) - 1$

Find values for a, b and c.

CHECK: $y = 3\cdot\sin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) - 1$

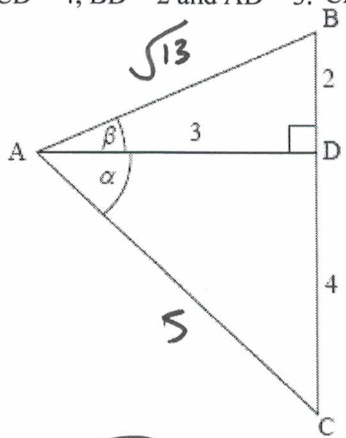
$y = 3\sin\left(\frac{\pi}{2}\right) - 1$

$y = 3 - 1$

$y = 2$ ✓

(Total 6 marks)

35. In the diagram below, AD is perpendicular to BC.
CD = 4, BD = 2 and AD = 3. $C\hat{A}D = \alpha$ and $B\hat{A}D = \beta$.



$\sin B = \frac{2}{\sqrt{13}}$

$\cos B = \frac{3}{\sqrt{13}}$

$\sin a = \frac{4}{5}$

$\cos a = \frac{3}{5}$

$\cos(a - B) = \cos a \cos B + \sin a \sin B$

$= \left(\frac{3}{5}\right)\left(\frac{3}{\sqrt{13}}\right) + \left(\frac{4}{5}\right)\left(\frac{2}{\sqrt{13}}\right)$

$= \frac{9}{5\sqrt{13}} + \frac{8}{5\sqrt{13}}$

$= \frac{17}{5\sqrt{13}}$ or $\frac{17\sqrt{13}}{65}$

Find the exact value of cos ($\alpha - \beta$).

NO DECIMALS!!

(Total 6 marks)

---

36. Verify the identity $\frac{\cot^2\theta}{1+\csc\theta} = \frac{1-\sin\theta}{\sin\theta}$.

START FROM MORE COMPLICATED SIDE

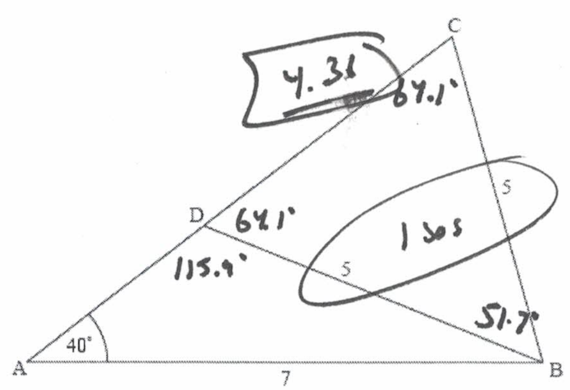
$\frac{\csc^2\theta - 1}{1+\csc\theta} = \frac{(\csc\theta - 1)\cancel{(\csc\theta + 1)}}{\cancel{\csc\theta + 1}} = \csc\theta - 1 = \frac{1}{\sin\theta} - 1 = \frac{1}{\sin\theta} - \frac{\sin\theta}{\sin\theta}$

$= \boxed{\frac{1-\sin\theta}{\sin\theta}}$ ✓

(Total 6 marks)

---

37. Given ΔABC, with lengths shown in the diagram below, find the length of the line segment [CD].
(*diagram not to scale)*



LAW OF SINES

$\frac{\sin 40}{5} = \frac{\sin x}{7}$

$x = \sin^{-1}\left(\frac{7 \cdot \sin 40}{5}\right)$

$x = 64.1°$ or $115.9°$

MUST BE 115.9

SINCE ΔBDC ISOS. AND

$\angle C \neq 115.9°$

$\angle B = 180 - 2 \cdot 64.1$

$\angle B = 51.7$

$\frac{\sin 64.1}{5} = \frac{\sin 51.7}{y}$

$y = \frac{5 \cdot \sin 51.7}{\sin 64.1}$

$\boxed{y \approx 4.36}$ CD

(Total 5 marks)

38. A fair six-sided die, with sides numbered 1, 1, 2, 3, 4, 5 is thrown. Find the mean and variance of the score.

$$\text{MEAN} = \frac{1+1+2+3+4+5}{6} = \frac{16}{6} = \boxed{\frac{8}{3}}$$

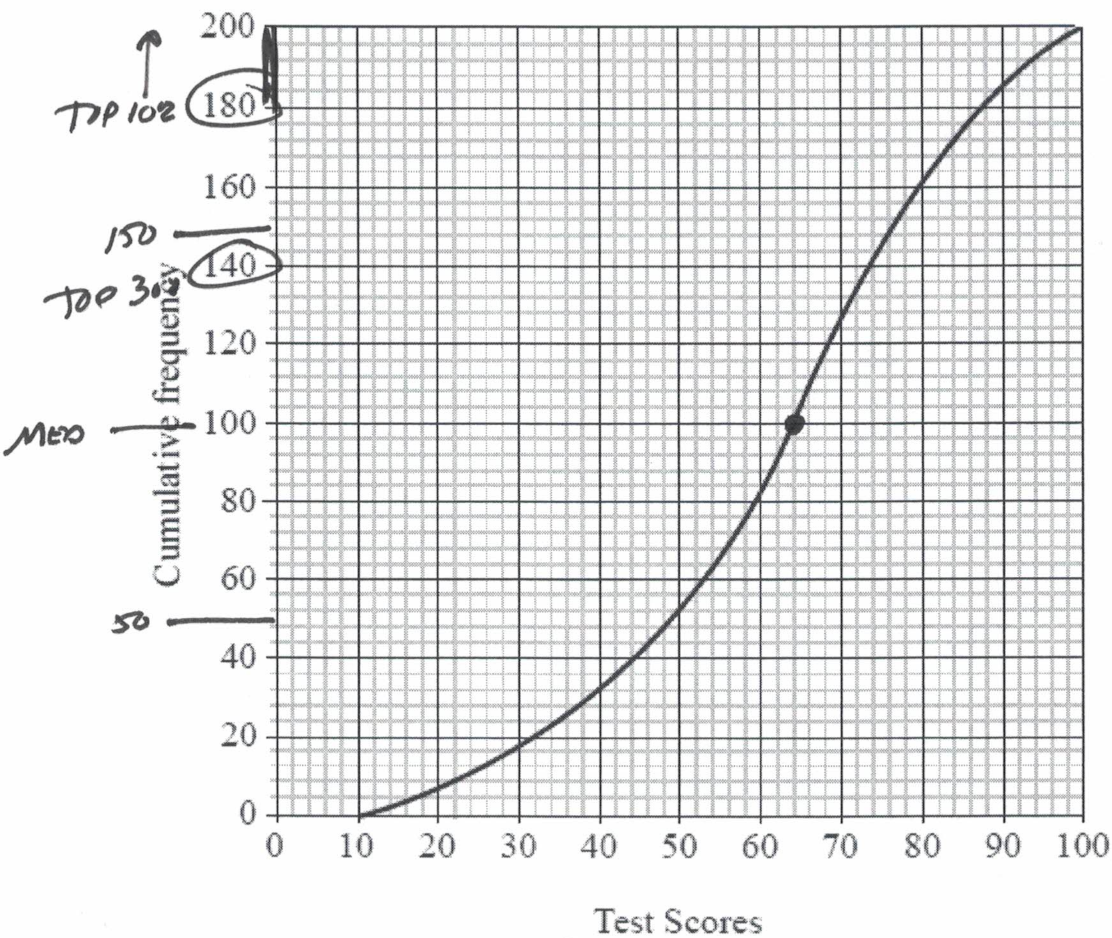
$$\text{VARIANCE} = \frac{1^2+1^2+2^2+3^2+4^2+5^2}{6} - \left(\frac{8}{3}\right)^2 = \boxed{\frac{20}{9}}$$

(VARIANCE = MEAN OF SQUARES − SQUARE OF MEANS)

(Total 6 marks)

---

39. The test scores of a group of students are shown on the cumulative frequency graph below.



(a) Estimate the median test score. $\boxed{64}$ **(1)**

(b) The top 10 % of students receive a grade A and the next best 20 % of students receive a grade B.

Estimate

TOP 20 TOP 60

(i) the minimum score required to obtain a grade A; $\boxed{87}$

(ii) the minimum score required to obtain a grade B. $\boxed{74}$

±1 IS OK!

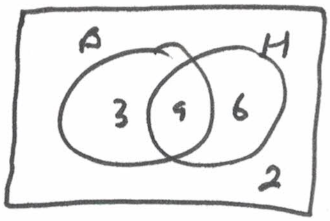
(4)

(Total 5 marks)

40. In a class of 20 students, 12 study Biology, 15 study History and 2 students study neither Biology nor History.

(a) Illustrate this information on a Venn diagram. **(2)**

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



X = # TAKING BOTH

$20 = 12 + 15 - x + 2$

$\boxed{x = 9}$

(b) Find the probability that a randomly selected student from this class is studying both Biology and History. **(1)**

$P(B \cap H) = \frac{9}{20}$ or $\boxed{0.45}$

(c) Given that a randomly selected student studies Biology, find the probability that this student also studies History. **(1)**

CONDITIONAL PROB

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(H|B) = \frac{P(H \cap B)}{P(B)}$

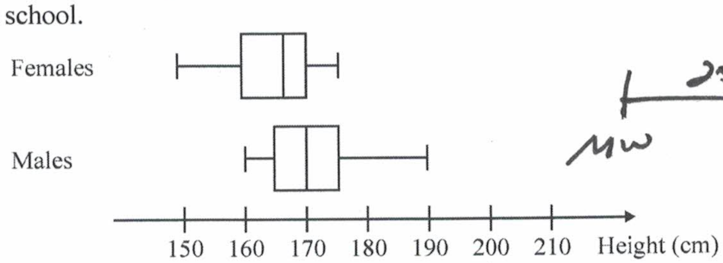
$= \frac{9/20}{12/20}$

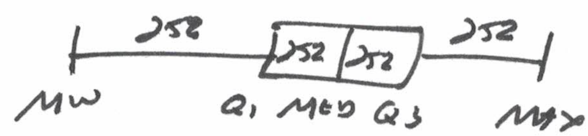
$\boxed{= \frac{3}{4} \text{ or } 0.75}$

(Total 4 marks)

---

41. The box-and-whisker plots shown represent the heights of female students and the heights of male students at a certain school.





(a) What percentage of female students are shorter than any male students?

25%

(b) What percentage of male students are shorter than some female students?

75%

(c) From the diagram, **estimate** the mean height of the male students.

TOP HEAVY, $\boxed{>170}$ (171–173) NOT 170!

(Total 3 marks)

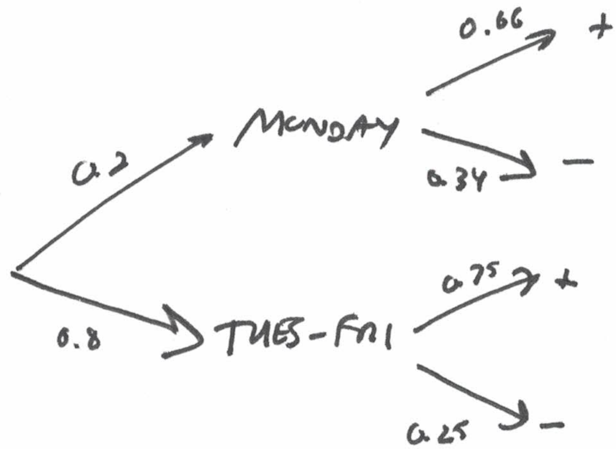
42. Robert travels to work by train every weekday from Monday to Friday. The probability that he catches the 08.00 train on Monday is 0.66. The probability that he catches the 08.00 train on any other weekday is 0.75. A weekday is chosen at random.

(a) Find the probability that he catches the train on that day.

(b) Given that he catches the 08.00 train on that day, find the probability that the chosen day is Monday.

+ = CATCHES TRAIN

− = MISSES TRAIN



(a) $P(+) = 0.2 \cdot 0.66 + 0.8 \cdot 0.75$

$= \boxed{0.732}$

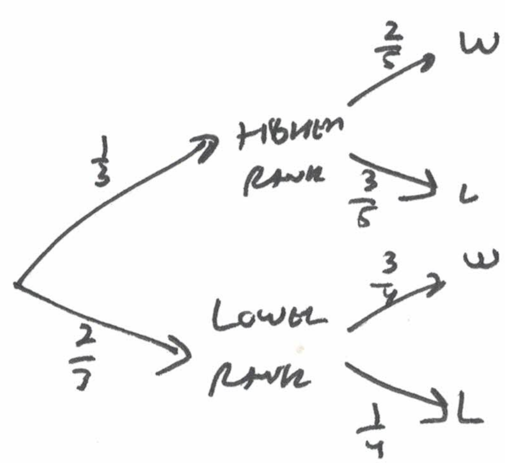
(b) $P(M|+) = \dfrac{P(M \cap +)}{P(+)} = \dfrac{0.2 \cdot 0.66}{0.732}$

$= \boxed{0.180}$

(Total 6 marks)

---

43. The local Football Association consists of ten teams. Team *A* has a 40% chance of winning any game against a higher-ranked team, and a 75% chance of winning any game against a lower-ranked team. If *A* is currently in fourth position, find the probability that *A* wins its next game.



1 2 3 $\boxed{5}$ 5 6 7 8 9 10

3 TEAMS

6 TEAMS

$P(W) = \frac{1}{3} \cdot \frac{2}{5} + \frac{2}{3} \cdot \frac{3}{4}$

$P(W) = \frac{2}{15} + \frac{1}{2}$

$\boxed{P(W) = \frac{19}{30} \text{ or } 0.6\overline{3}}$

(Total 4 marks)

44. Use mathematical induction to prove that $1+2+3+...+n=\dfrac{n(n+1)}{2}$ for $n \in \mathbb{Z}^+$.

① PROVE P(1) TRUE

$1 = \frac{1(1+1)}{2}$

$1 = 1$ ✓

② ASSUME P(k) TRUE

$1+2+...+k = \frac{k(k+1)}{2}$

③ PROVE P(k+1) TRUE

$1+2+...+k+(k+1) = \frac{(k+1)(k+2)}{2}$

from ②

$1+2+...+k = \frac{k(k+1)}{2}$

$1+2+...+k+k+1 = \frac{k(k+1)}{2} + k+1$

$= \frac{k^2+k+2k+2}{2}$

$= \frac{k^2+3k+2}{2}$

$= \frac{(k+1)(k+2)}{2}$ ✓

---

45. Use mathematical induction to prove $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+...+\dfrac{1}{2^n}=1-\left(\dfrac{1}{2}\right)^n$

① $\frac{1}{2} = 1-\left(\frac{1}{2}\right)^1$

$\frac{1}{2} = \frac{1}{2}$ ✓

② $\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^k} = 1-\left(\frac{1}{2}\right)^k$

③ $\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^k}+\frac{1}{2^{k+1}} = 1-\left(\frac{1}{2}\right)^{k+1}$

$\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^k} = 1-\left(\frac{1}{2}\right)^k$

$\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^k}+\frac{1}{2^{k+1}} = 1-\left(\frac{1}{2}\right)^k+\frac{1}{2^{k+1}}$

$= 1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$

$= 1-\frac{2}{2^{k+1}}+\frac{1}{2^{k+1}}$

$= 1-\frac{1}{2^{k+1}}$

$= 1-\left(\frac{1}{2}\right)^{k+1}$ ✓

---

46. Use mathematical induction to prove that $4^{2n}-1$ is divisible by 5, for $n \in \mathbb{Z}^+$.

① $4^2-1 = 5A$ $\quad A, B, C \in \mathbb{Z}$

$15 = 5A$ $\underline{A=3}$ ✓

② $4^{2k}-1 = 5B$

③ $4^{2k+2}-1 = 5C$

$4^{2k} \cdot 4^2 - 1 = 5C$

$\left(4^{2k}-1 = 5B\right) \cdot 4^2$

$4^2 \cdot 4^{2k} - 1 \cdot 4^2 = 5B \cdot 4^2$

$4^{2k+2} - 1 - 15 = 80B$

$4^{2k+2} - 1 = 80B + 15$

$4^{2k+2} - 1 = 5(16B+3)$

$4^{2k+2} - 1 = 5C$ ✓

47. The complex number z satisfies $i(z + 2) = 1 - 2z$, where $i = \sqrt{-1}$. Write z in the form $z = a + bi$, where a and b are real numbers.

$iz + 2i = 1 - 2z$

$2z + iz = 1 - 2i$

$z(2+i) = 1 - 2i$

$z = \frac{1-2i}{2+i}$

$z = \frac{(1-2i)(2-i)}{(2+i)(2-i)}$

$z = \frac{2 - i - 4i + 2i^2}{4 - i^2}$

$z = -i$

$z = -i$

$\boxed{z = 0 - 1i}$

(Total 3 marks)

48. Given that $(a + bi)^2 = 3 + 4i$ obtain a pair of simultaneous equations involving a and b. Hence find the two square roots of $3 + 4i$.

$a^2 + 2abi + b^2 i^2 = 3 + 4i$

$a^2 - b^2 + 2abi = 3 + 4i$

$a^2 - b^2 = 3$

$2ab = 4$

$b = \frac{2}{a}$

$a^2 - \left(\frac{2}{a}\right)^2 = 3$

$\left(a^2 - \frac{4}{a^2} = 3\right) \cdot a^2$

$a^4 - 3a^2 - 4 = 0$

$(a^2 - 4)(a^2 + 1) = 0$

$a = \pm 2$ NO SOLUTION

$b = \pm 1$

$\boxed{\begin{matrix} 2+i \\ -2-i \end{matrix}}$

(Total 7 marks)

49. Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$. Given that $1 + i$ and $1 - 2i$ are zeros of $p(x)$, find the values of a, b, c and d.

CONJUGATE ROOT THM

ROOTS: $1 \pm i$, $1 \pm 2i$

$x = 1+i \quad x = 1-i \quad x = 1+2i \quad x = 1-2i$

$(x-1-i)(x-1+i)(x-1-2i)(x-1+2i)$

$((x-1)^2 - i^2)((x-1)^2 - 4i^2)$

$(x^2 - 2x + 1 + 1)(x^2 - 2x + 1 + 4)$

$(x^2 - 2x + 2)(x^2 - 2x + 5)$

$x^4 - 2x^3 + 5x^2$

$-2x^3 + 4x^2 - 10x$

$2x^2 - 4x + 10$

$\boxed{P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10}$

(Total 7 marks)

50. The polynomial $f(x) = x^2 + 9x + 33$ has roots of a and b. Without finding the actual roots, find the exact value of the expression $\frac{1}{a^2} + \frac{1}{b^2}$ using Viete's Theorem. Leave your answer as a reduced fraction.

VIETE'S THM

SUM $= a + b = -\frac{b}{a} = -\frac{9}{1} = -9$

PRODUCT $= a \cdot b = \frac{c}{a} = \frac{33}{1} = 33$

$\frac{1}{a^2} + \frac{1}{b^2} = \frac{b^2 + a^2}{a^2 b^2}$

$= \frac{a^2 + 2ab + b^2 - 2ab}{a^2 b^2}$

$= \frac{(a+b)^2 - 2(ab)}{(ab)^2}$

$= \frac{(-9)^2 - 2(3\ldots}{(33)^2}$

$= \frac{15}{1089}$

$= \boxed{\frac{5}{363}}$

(Total 7 marks)

51. The cubic equation $f(x) = 3x^3 + x^2 - 6x + 20$ has roots of x, y and z. Without finding the actual roots, find the exact value of the expression $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ using Viete's Theorem. Leave your answer as a reduced fraction.

$x+y+z = -\frac{b}{a} = -\frac{1}{3}$

$xy+yz+xz = \frac{c}{a} = -2$

$xyz = -\frac{d}{a} = -\frac{20}{3}$

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{yz+xz+xy}{xyz}$

$= \frac{-2}{\frac{-20}{3}}$

$= -2 \cdot -\frac{3}{20}$

$= \boxed{\frac{3}{10}}$

(Total 6 marks)

---

52. (a) Show that the complex number i is a root of the equation

$$x^4 - 5x^3 + 7x^2 - 5x + 6 = 0.$$

(2)

i	1	-5	7	-5	6
	↓	i	$-1-5i$	$5+6i$	-6
	1	$-5+i$	$6-5i$	$6i$	$\boxed{0}$

✓

(b) Find the other roots of this equation.

(4)

CONJUGATE ROOT THM

$-i$	1	$-5+i$	$6-5i$	$6i$
	↓	$-i$	$5i$	$-6i$
	1	-5	6	$\boxed{0}$

$x^2 - 5x + 6 = (x-3)(x-2)$

$\boxed{\text{ROOTS: } \pm i, 2, 3}$

(Total 6 marks)

---

53. Given that $\frac{z}{z+2} = 2 - \text{i}$, $z \in \mathbb{C}$, find z in the form $a + \text{i}b$.

$z = (2-i)(z+2)$

$z = 2z + 4 - iz - 2i$

$-1z + iz = 4 - 2i$

$z(-1+i) = 4 - 2i$

$z = \frac{4-2i}{-1+i} \cdot \frac{(-1-i)}{(-1-i)}$

$z = \frac{-4-4i+2i+2i^2}{1-i^2}$

$z = \frac{-6-2i}{2}$

$\boxed{z = -3 - i}$

(Total 4 marks)

REMAINDER = 0

54. When $f(x) = x^4 + 3x^3 + px^2 - 2x + q$ is divided by $(x - 2)$ the remainder is 15, and $(x + 3)$ is a factor of $f(x)$. Find the values of p and q.

$$\begin{array}{r|rrrrr} 2 & 1 & 3 & p & -2 & q \\ & \downarrow & 2 & 10 & 2p+20 & 4p+36 \\ \hline & 1 & 5 & p+10 & 2p+18 & \boxed{15} \end{array}$$

$q + 4p + 36 = 15$

$4p + q = -21$

$$\begin{array}{r|rrrrr} -3 & 1 & 3 & p & -2 & q \\ & \downarrow & -3 & 0 & -3p & 9p+6 \\ \hline & 1 & 0 & p & -3p-2 & \boxed{0} \end{array}$$

$q + 9p + 6 = 0$

$9p + q = -6$

$4p + q = -21$

$-9p - q = 6$

$-5p = -15$

$\boxed{p = 3 \quad q = -33}$

(Total 6 marks)

---

55. (a) Evaluate $(1 + i)^2$, where $i = \sqrt{-1}$. **(2)**

$1 + 2i + i^2 = 1 + 2i - 1 = \boxed{2i}$

(b) Show that $(1 + i)^{4n} = (-4)^n$, where $n \in \mathbb{N}$. **(4)**

$[(1+i)^4]^n = [((1+i)^2)^2]^n = [(2i)^2]^n = (4i^2)^n = \boxed{(-4)^n}$

(c) Hence or otherwise, find $(1 + i)^{32}$. **(2)**

$(1+i)^{32} = (1+i)^{4 \cdot 8} = (-4)^8 = \boxed{65,536}$

(Total 8 marks)

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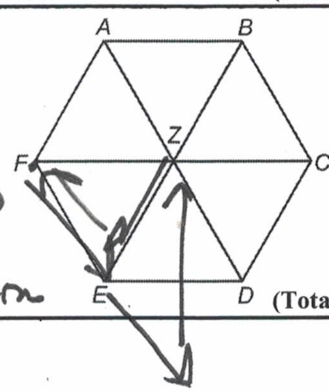
56. Use the regular hexagon to the right to answer parts (a) and (b).



a.) List **all** equivalent vectors to $-\frac{1}{2}\overrightarrow{CF}$. $= \frac{1}{2}\overrightarrow{FC}$

$\overrightarrow{FZ}, \overrightarrow{ZC}, \overrightarrow{AB}, \overrightarrow{ED}$

b.) Write an equivalent vector to $2\overrightarrow{FE} + \overrightarrow{DB} - \overrightarrow{ZB} + \overrightarrow{ZA}$. $= \boxed{0}$

$+ \overrightarrow{BZ}$

NULL VECTOR

(Total 5 marks)

---

57. The three vectors a, b and c are given by

$$a = \begin{pmatrix} 2y \\ -3x \end{pmatrix}, b = \begin{pmatrix} 4x \\ y \end{pmatrix}, c = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \text{ where } x, y \in \mathbb{R}.$$

(a) If $a + 2b - c = 0$, find the value of x and of y. **(3)**

$\begin{pmatrix} 2y \\ -3x \end{pmatrix} + \begin{pmatrix} 8x \\ 2y \end{pmatrix} + \begin{pmatrix} -4 \\ 7 \end{pmatrix} = 0$

$2y + 8x - 4 = 0$

$-3x + 2y + 7 = 0$

$8x + 2y = 4$

$-3x + 2y = -7$

$8x + 2y = 4$

$3x - 2y = 7$

$11x = 11$

$\boxed{x = 1 \quad y = -2}$

(b) Find the exact value of $|a + 2b|$. **(2)**

$a = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ $b = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ $2b = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$ $a + 2b = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$

(Total 5 marks)

$|a + 2b| = \sqrt{16 + 49} = \boxed{\sqrt{65}}$