Math HL – Year 2
Calculus Option – Paper 3 Review

Name: KEY

Period: _____

Evaluate, if possible, the following limits. Use ∞ & $-\infty$ as necessary and show all work.

1.) $\lim_{x\to 2}\dfrac{\frac{1}{2+x}-\frac{1}{4}}{2-x} \cdot \dfrac{4(2+x)}{4(2+x)}$

$= \dfrac{4-(2+x)}{4(2+x)(2-x)}$

$= \dfrac{\cancel{2-x}}{4(2+x)\cancel{(2-x)}}$

$= \dfrac{1}{4(2+x)}$

$\boxed{= \dfrac{1}{16}}$

2.) $\lim_{x\to 0}\dfrac{(x-2)^3+8}{x} = \dfrac{x^3-6x^2+12x-8+8}{x}$

$(x-2)(x-2)(x-2)$

$(x^2-4x+4)(x-2)$

$x^3-4x^2-2x^2+4x+8x-8$

$x^3-6x^2+12x-8$

$= \dfrac{x^3-6x^2+12x}{x}$

$= x^2-6x+12$

$\boxed{= 12}$

3.) $\lim_{x\to\infty}\dfrac{7x^3+3x+6}{5x^5-4x^3-2} \cdot \dfrac{\frac{1}{x^5}}{\frac{1}{x^5}} = \dfrac{\frac{7}{x^2}+\frac{3}{x^4}+\frac{6}{x^5}}{5-\frac{4}{x^2}-\frac{2}{x^5}}$

$= \dfrac{0+0+0}{5+0+0}$

$\boxed{= 0}$

4.) $\lim_{x\to 5}\dfrac{\sqrt{x+20}-5}{x-5} \cdot \dfrac{\sqrt{x+20}+5}{\sqrt{x+20}+5}$

$= \dfrac{(x+20)-25}{(x-5)(\sqrt{x+20}+5)}$

$= \dfrac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x+20}+5)}$

$= \dfrac{1}{\sqrt{x+20}+5}$

$\boxed{= \dfrac{1}{10}}$

5.) $\lim_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right) = \lim_{x\to 0}\dfrac{\sin x - x}{x\sin x} \quad \dfrac{0}{0}$

$= \lim_{x\to 0}\dfrac{\cos x - 1}{\sin x + x\cos x} \quad \dfrac{0}{0}$

$= \lim_{x\to 0}\dfrac{-\sin x}{\cos x + \cos x + x(-\sin x)}$

$= \dfrac{0}{1+1+0}$

$= \dfrac{0}{2}$

$\boxed{= 0}$

6.) $\lim_{x\to 0} x^2\cos\left(\dfrac{1}{x}\right)$

$-1 \le \cos\frac{1}{x} \le 1$

$-x^2 \le x^2\cdot\cos\frac{1}{x} \le x^2$

$\lim_{x\to 0} -x^2 = 0$

$\lim_{x\to 0} x^2 = 0$

$\therefore$ BY SQUEEZE THM,

$\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) \boxed{= 0}$

7.) Use the definition of continuity to prove if the function below is continuous at $x = -4$.

No graphical explanations!

$$f(x)=\begin{cases}\dfrac{x^2+2x-8}{x+4} & \text{if } x \neq -4 \\ -3 & \text{if } x = -4\end{cases}$$

1.) $f(-4) = -3$

2.) $\frac{x^2+2x-8}{x+4} = \frac{(x+4)(x-2)}{x+4} = x-2$

$\lim_{x \to -4} = -6$

3.) $-3 \neq -6$

NOT CONTINUOUS!!

CONTINUOUS

1.) $f(x)$ EXISTS AT C

2.) $\lim_{x \to c} f(x)$ EXISTS (LHL = RHL)

3.) $\lim_{x \to c} f(x) = f(c)$ (#1 + 2 EQUAL)

8.) a.) Sketch the function:

$$f(x)=\begin{cases}-2x-3 & x<-1 \\ x^2-2 & -1\le x<2 \\ 2 & x\ge 2\end{cases}$$

$f(-1) = -1$

$f(-1) = -1$ $f(2) = 2$

$f(2) = 2$

b.) Determine and prove where the function is continuous.

CONT.

1.) ✓ $f(-1) = -1$ $f(2) = 2$

2.) $\lim_{x \to -1} = -1$ $\lim_{x \to 2} = 2$ ✓

3.) ✓ 1.) = 2.)

$\boxed{\text{CONTINUOUS FOR } x \in \mathbb{R}.}$

c.) Determine and prove where the function is differentiable.

DIFF.

1.) CONTINUOUS ✓ AT $x = -1, 2$

2.) NO VERTICAL TANG. ✓

3.) NO "SHARP" TURNS

$f(x) = \begin{cases} -2x-3 \\ x^2-2 \\ 2 \end{cases}$ $f'(x) = \begin{cases} -2 \\ 2x \\ 0 \end{cases}$

AT $x=-1$

$f'(-1) = -2$

$f'(-1) = -2$ ✓

AT $x = 2$

$f'(2) = 4$

$f'(2) = 0$

$4 \neq 0$

$\boxed{\text{DIFF AT } x \in \mathbb{R},\ x \neq 2.}$

Evaluate, if possible, the following limits. Use ∞ & – ∞ as necessary and show all work.

9.) $\lim_{x \to -1} \begin{cases} \dfrac{x^2-3x-4}{x+1} & x \neq -1 \\ 2 & x = -1 \end{cases}$

$\dfrac{x^2-3x-4}{x+1} = \dfrac{(x-4)(x+1)}{x+1} = x-4$

$\lim_{x \to -1} (x-4) \boxed{= -5}$

10.) $\lim_{x \to \infty} \dfrac{x^2}{e^x}$ $\left(\dfrac{\infty}{\infty}\right) = \lim_{x \to \infty} \dfrac{2x}{e^x} \left(\dfrac{\infty}{\infty}\right)$

$= \lim_{x \to \infty} \dfrac{2}{e^x}$

$\boxed{= 0}$

11.) $\lim_{h \to 0} \dfrac{(x+h)^3 - (x^3)}{h}$ (in terms of x)

$= \lim_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$

$= \lim_{h \to 0} 3x^2 + \cancel{3xh} + \cancel{h^2}$

$\boxed{= 3x^2}$

12.) $\lim_{h \to 0} \dfrac{\cos\left(\frac{\pi}{2} + h\right)}{h}$ $\left(\dfrac{0}{0}\right) = \dfrac{-\sin\left(\frac{\pi}{2}+h\right)}{1}$

$= \dfrac{-1}{1}$

$\boxed{= -1}$

13.) ***Multiple Choice. Show work for full credit.***

Which of the following is/are true about the function g if $g(x) = \dfrac{(x-2)^2}{x^2+x-6}$? $= \dfrac{x-2}{x+3}$

$(x+3)(x-2)$

REMOVEABLE

~~I.~~ g is continuous at $x = 2$

II. The graph of g has a vertical asymptote at $x = -3$ YES!

~~III.~~ The graph of g has a horizontal asymptote at $y = 0$ POWERS EQUAL, ASYM @ $y = 1$

a.) I only

b.) II only

c.) III only

d.) I and II only

e.) II and III only

X NOT CONTINUOUS AT $x = 2$ ($(x-2)$ IN DENOM)

✓. VERT ASYMP. AT $x = -3$ ($(x+3)$ IN DENOM)

X. POWERS EQUAL (DEGREE = 2), DIVIDE COEFF., HA @ $y = 1$

14.) ***Multiple Choice.*** ***You must include work to justify your answer!!***

At $x=4$, the function given by $h(x)=\begin{cases} x^2 & x\le 4 \\ 4x & x>4 \end{cases}$ is

~~(A) Undefined~~
(B) Continuous but not differentiable AT $x=4$
(C) Differentiable but not continuous
~~(D) Neither continuous nor differentiable~~
~~(E) Both continuous and differentiable~~

$(4)^2 = 16$
$4(4) = 16$ CONTINUOUS

$h'(x) = \begin{cases} 2x \\ 4 \end{cases}$ $2(4) = 8$ 4 $8 \neq 4$ SHARP TURN

15.) Sketch the graph of ONE function *f (x)* that satisfies all of the following conditions.

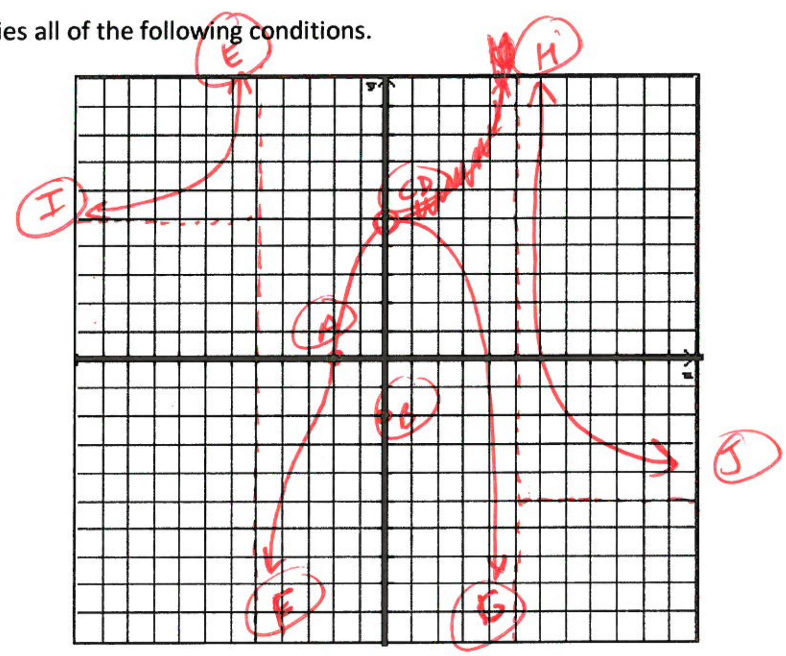
(A) $f(-2)=0$ (B) $f(0)=-2$

(C) $\lim_{x\to 0-} f(x)=5$ (D) $\lim_{x\to 0+} f(x)=5$

(E) $\lim_{x\to -5-} f(x)=\infty$ (F) $\lim_{x\to -5+} f(x)=-\infty$

(G) $\lim_{x\to 5-} f(x)=-\infty$ (H) $\lim_{x\to 5+} f(x)=\infty$

(I) $\lim_{x\to -\infty} f(x)=5$ (J) $\lim_{x\to \infty} f(x)=-5$

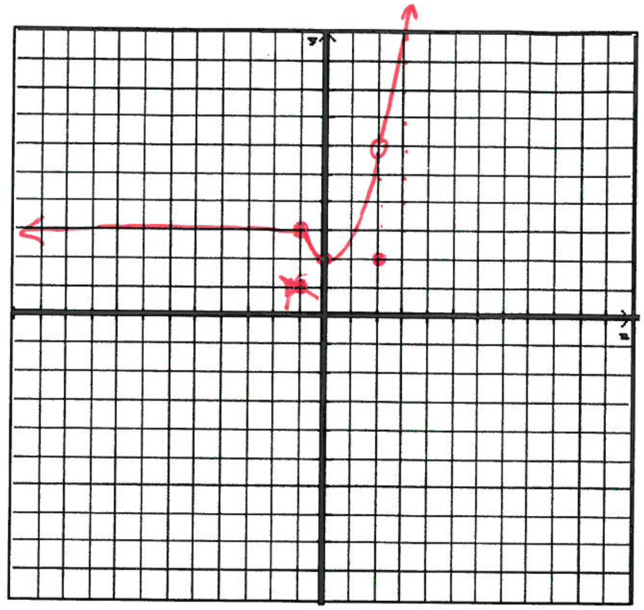


16.) a.) Sketch the function:

$$f(x)=\begin{cases}3 & x<-1\\ x^2+2 & -1\le x<2\\ 2 & x=2\\ |4x-2| & x>2\end{cases}$$

$f(-1)=3$ $f(2)=6$

$f(2)=6$



b.) Determine and prove where the function is continuous.

AT $x=-1$

1.) $f(-1)=3$ ✓

2.) $\lim_{x\to -1} = 3$ ✓

3.) $3=3$ ✓

CONT.

AT $x=2$

1.) $f(2)=2$

2.) $\lim_{x\to 2} = 6$

3.) $2\neq 6$

NOT CONT.

CONTINUOUS

$x\in\mathbb{R},\ x\neq 2$

c.) Determine and prove where the function is differentiable.

AT $x=-1$

1.) CONTIN. ✓

2.) NO VERT. TANG ✓

3.) $f'(-1)=0$, $f'(-1)=2(-1)=-2$; $0\neq -2$ SHARP TURN

NOT DIFF.

AT $x=2$

X 1.) NOT CONT

DIFFERENTIABLE

$x\in\mathbb{R},\ x\neq -1, 2$

17.) Prove the equation $x^3+9x^2+28x+30=0$ has only 1 real root.

CHECK POINTS ON GRAPH

AT $x=-3$, $f(x)=0$ ✓

1 ROOT.

LET r = A 2ND ROOT OF $f(x)$. $f(-3)=f(r)=0$ (Y-INT.).

BY ROLLE'S THM, IF $f(-3)=f(r)$, THEN THERE EXISTS SOME VALUE C SUCH THAT $f'(c)=0$. $f'(x)=3x^2+18x+28$. USING THE QUADRATIC FORMULA, $3x^2+18x+28=0$ GIVES THE SOLUTIONS $\frac{-18\pm\sqrt{-12}}{6}$, GIVING NO REAL SOLUTIONS. THUS, $f'(x)\neq 0$ ANYWHERE $x\in\mathbb{R}$. $\therefore$ THIS POLYNOMIAL FAILS ROLLE'S THM, SO r DOES NOT EXIST, GIVING YOU 1 REAL ROOT AT $x=-3$.

18.) The function $f(x)$ is define for $x \in \left]3,10\right[$ and has $f(3) = -2$ and $f'(x) \geq 2$.
Use the Mean Value Theorem to find the smallest possible value for $f(10)$.

$$\frac{f(b)-f(a)}{b-a} = f'(c) \qquad f(3) = -2, \quad f(10) = ?$$

$$\frac{f(10)-f(3)}{10-3} = f'(c)$$

$$\frac{f(10)+2}{7} = f'(c)$$

$$f(10) = 7 \cdot f'(c) - 2$$

$$f(10) = 7 \cdot 2 - 2$$

$$f(10) = 12$$

12 IS MIN VALUE OF $f(10)$.

19.) Determine whether the ~~series~~ INTEGRAL converges or diverges. If it converges, evaluate the integral (Exact value).

a.) $\displaystyle\int_1^\infty \frac{6x^2}{x^3-1}\,dx = 2\int_1^\infty \frac{1}{u}\cdot du$

$u = x^3 - 1$

$du = 3x^2 \cdot dx$

$2 \cdot du = 6x^2 \cdot dx$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{u} \cdot du$$

$$= \lim_{b\to\infty} \left[\ln|u|\right]_1^b$$

$$= \lim_{b\to\infty} \left[\ln|x^3-1|\right]_1^b$$

$$= \lim_{b\to\infty} \ln|b^3-1| -$$

b.) $\displaystyle\int_2^\infty 4e^{-x}\,dx = \lim_{b\to\infty} \int_2^b 4e^{-x} \cdot dx$

$$= \text{"} \left[-4e^{-x}\right]_2^b$$

$$= \text{"} \left(\frac{-4}{e^b} - \frac{-4}{e^2}\right)$$

$$\boxed{= \frac{4}{e^2}}$$ CONVERGES

20.) Determine whether the ~~series~~ INTEGRAL converges or diverges.

a.) $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{2n+1}}\,dx$

$$\frac{1}{\sqrt[3]{2n+1}} < \underbrace{\frac{1}{\sqrt[3]{n}}}_{\text{DIVERGES}}$$

INCONCLUSIVE

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n}}}{\frac{1}{\sqrt[3]{2n+1}}} = \frac{\sqrt[3]{2n+1}}{\sqrt[3]{n}} = \sqrt[3]{\frac{2n+1}{n}} = \sqrt[3]{2+\frac{1}{n}} = \sqrt[3]{2} > 0$$

WORK TOGETHER.

SINCE $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{n}}$ DIVERGES BY P-SERIES TEST, $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{2n}} \cdot dx$ DIVERGES BY LIMIT COMPARISON TEST.

b.) $\displaystyle\int_0^\infty \frac{\cos^2 x}{2x^3} \cdot dx$

$0 \leq \cos^2 x \leq 1$

$$\frac{\cos^2 x}{2x^3} \leq \frac{1}{2x^3} \leq \underbrace{\frac{1}{x^3}}_{}$$

CONVERGES BY P-SERIES TEST.

$\therefore \displaystyle\int \frac{\cos^2 x}{2x^3} \cdot dx$ CONVERGES BY COMPARISON TEST.

21.) Evaluate the following convergent integral, $\int_1^{\infty} \frac{1}{(1-5x)^2} \cdot dx$ (Exact value).

$$\int_1^{\infty} (1-5x)^{-2}\,dx = \lim_{b\to\infty} \left[-1 \cdot -\frac{1}{5}(1-5x)^{-1}\right]_1^b$$

$$\lim_{b\to\infty} \left[\frac{1}{5}(1-5x)^{-1}\right]_1^b$$

$$\frac{1}{5} \cdot \lim_{b\to\infty} \left(\underbrace{\cancel{\frac{1}{1-5b}}}_{0} - \frac{1}{1-5}\right)$$

$$\frac{1}{5} \cdot \frac{1}{4}$$

$$\boxed{\frac{1}{20}}$$

22.) a.) Using summation notation (you do not have to evaluate), find L and U such that:

DECREASING FXN

$$L < \int_n^{\infty} \frac{1}{x^3} \cdot dx < U$$

$$\underbrace{\boxed{\sum_{n+1}^{\infty} \frac{1}{x^3}}}_{L} < \int_n^{\infty} \frac{1}{x^3} \cdot dx < \underbrace{\boxed{\sum_{n}^{\infty} \frac{1}{x^3}}}_{U}$$

b.) In terms of n, evaluate the integral $\int_n^{\infty} \frac{1}{x^3} \cdot dx$.

$$\int_n^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_n^b \frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{2}x^{-2}\right]_n^b$$

$$= \lim_{b\to\infty} -\frac{1}{2}\left(\cancel{\frac{1}{b^2}} - \frac{1}{n^2}\right)$$

$$\boxed{= \frac{1}{2n^2}}$$

c.) Hence or otherwise, find the area under the curve $y = \frac{1}{x^3}$ on the interval $[1,5]$.

$$A = \int_1^5 \frac{1}{x^3} \cdot dx = \int_1^{\infty} \frac{1}{x^3}\,dx - \int_5^{\infty} \frac{1}{x^3}\,dx$$

$$= \frac{1}{2(1)^2} - \frac{1}{2(5)^2}$$

$$= \frac{1}{2} - \frac{1}{50}$$

$$= \frac{24}{50} \text{ or } \boxed{\frac{12}{25}}$$

$$\int_1^5 \frac{1}{x^3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^5$$

$$= -\frac{1}{2}\left(5^{-2} - 1^{-2}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{25} - 1\right)$$

$$= -\frac{1}{\cancel{2}} \cdot -\frac{\cancel{24}^{12}}{25}$$

$$\boxed{= \frac{12}{25}}$$

23.) Use the comparison test to determine whether the following integral converges or diverges.

$$\int_1^\infty \frac{4}{\sqrt{x}-7}\,dx$$

$$\underline{\frac{4}{\sqrt{x}} < \frac{4}{\sqrt{x}-7}}$$

$4\int \frac{1}{\sqrt{x}}\,dx$ diverges by P-SERIES TEST.

THEREFORE, BY THE COMPARISON TEST $\int_1^\infty \frac{4}{\sqrt{x}-7}\,dx$ DIVERGES.

24.) Find the upper and lower sums of the integral $\int_1^\infty 5^{-x}\,dx$ (Summation notation only) DECREASING

$$\underbrace{\boxed{\sum_{x=2}^{\infty} 5^{-x}}}_{L} < \int_1^\infty 5^{-x}\,dx < \underbrace{\boxed{\sum_{x=1}^{\infty} 5^{-x}}}_{U}$$

Thus, find exact values for A and B such that $A < \int_1^\infty 5^{-x}\,dx < B$.

LOWER

$$\sum_{x=2}^{\infty} 5^{-x} = \sum_{x=2}^{\infty} \frac{1}{5^x}$$

$$= \underbrace{\sum_{x=2}^{\infty} \left(\frac{1}{5}\right)^x}_{\text{GEOMETRIC SERIES}}$$

$$S = \frac{a_1}{1-r}$$

$$= \frac{1/25}{1-1/5}$$

$$= \frac{1}{25-5}$$

$$= \frac{1}{20}$$

UPPER

$$\sum_{x=1}^{\infty} \left(\frac{1}{5}\right)^x$$

$$S = \frac{a_1}{1-r}$$

$$= \frac{1/5}{1-1/5}$$

$$= \frac{1}{5-1}$$

$$= \frac{1}{4}$$

$$\boxed{\underset{0.05}{\underline{\frac{1}{20}}} < \int_1^\infty 5^{-x} < \underset{0.25}{\underline{\frac{1}{4}}}}$$

Determine whether the following series converge or diverge. Show all work and clearly state which test(s) are used.

25.) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+6}}$ LIMIT COMPARISON TEST

$\lim_{k \to \infty} \frac{\frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k+6}}} = \frac{\sqrt{k+6}}{\sqrt{k}} = \sqrt{\frac{k+6}{k}} = \sqrt{1+\frac{6}{k}} = 1 > 0$ WORK TOGETHER

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ DIVERGES BY P-SERIES TEST, $\therefore \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+6}}$ DIVERGES BY THE LIMIT COMPARISON TEST.

26.) $\sum_{k=1}^{\infty} \frac{\sqrt{k^4+6}}{k^2+2}$ $\frac{\sqrt{k^4}}{k^2} = \frac{k^2}{k^2} = 1$

$\lim_{k \to \infty} \frac{\sqrt{k^4+6}}{k^2+2} \quad \frac{\frac{1}{\sqrt{k^4}}}{\frac{1}{\sqrt{k^4}}\left(\frac{1}{k^2}\right)} = \frac{\sqrt{\frac{k^4+6}{k^4}}}{\frac{k^2+2}{k^2}}$

$= \frac{\sqrt{1+\frac{6}{k^4}}}{1+\frac{2}{k^2}}$

$= \frac{1}{1}$

$= 1$ ✓

$\lim_{k \to \infty} \neq 0$, $\therefore$ DIVERGES BY DIVERGENCE TEST.

27.) $\sum_{k=1}^{\infty} \frac{k}{k^3+2k+4}$ $\frac{1}{k^2}$

$$\lim_{k\to\infty} \frac{\frac{1}{k^2}}{\frac{k}{k^3+2k+4}} = \frac{k^3+2k+4}{k^3} = 1+\cancel{\frac{2}{k^2}}+\cancel{\frac{4}{k^3}} = 1 > 0$$

WORK TOGETHER

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ CONVERGES BY P-SERIES TEST, $\therefore \sum_{k=1}^{\infty} \frac{k}{k^3+2k+4}$ CONVERGES

BY THE LIMIT COMPARISON TEST.

28.) $\sum_{k=2}^{\infty} \frac{k^2}{(2k-1)!}$ RATIO TEST

$$u_k = \frac{k^2}{(2k-1)!} \qquad u_{k+1} = \frac{(k+1)^2}{(2(k+1)-1)!} = \frac{(k+1)^2}{(2k+1)!}$$

$$\lim_{k\to\infty} \frac{\frac{(k+1)^2}{(2k+1)!}}{\frac{k^2}{(2k-1)!}} = \frac{(k+1)^2(2k-1)!}{k^2(2k+1)!} = \left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{(2k+1)(2k)}$$

$$= 1 \cdot 0$$

$$= 0$$

$\therefore \sum_{k=2}^{\infty} \frac{k^2}{(2k-1)!}$ CONVERGES BY RATIO TEST.

Complete the following steps for the series below:

- a • Find the first 4 partial sums.
- b • Determine whether the series converges or diverges.
- c • If the series converges, determine the sum of the series.

29.) $\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = \quad \frac{2^k \cdot 2^{-1}}{3^k} = \frac{1}{2}\left(\frac{2}{3}\right)^k$ GEOMETRIC SERIES

a.) $S_1 = \frac{1}{3}$

$S_2 = \frac{2}{9}$

$S_3 = \frac{4}{27}$

$S_4 = \frac{8}{81}$

b.) GEOM. SERIES

$r = \frac{2}{3} < 1$ CONVERGES

OR

RATIO TEST

$u_k = \frac{2^{k-1}}{3^k} \quad u_{k+1} = \frac{2^k}{3^{k+1}}$

$\lim_{k \to \infty} \frac{2^k \cdot 3^k}{2^{k-1} \cdot 3^{k+1}} = \left|\frac{2}{3}\right| < 1$ CONVERGES.

c.) $S = \frac{a_1}{1-r}$

$S = \frac{1/3}{1 - 2/3}$

$S = \frac{1/3}{1/3}$

$\boxed{S = 1}$

For problems (2) – (7), determine whether the following series converge or diverge. Show all work and clearly state which test(s) are used.

30.) $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$

$\frac{1}{5^n} = \left(\frac{1}{5}\right)^n$ GEOM. SERIES $|r| < 1$

$\frac{1}{2+5^n} < \frac{1}{5^n}$

$\sum_{n=1}^{\infty} \frac{1}{5^n}$ CONVERGES.

$\therefore$ BY COMPARISON TEST $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ CONVERGES.

31.) $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ $\cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{1}{5+\frac{4}{n^2}}$ $\lim_{n\to\infty} = \frac{1}{5} \neq 0$

DIVERGES BY DIVERGENCE TEST

32.) $\sum_{n=1}^{\infty} 2^{-n} \cdot 3^{n-1}$ $= \frac{3^{n-1}}{2^n} = \frac{3^n \cdot 3^{-1}}{2^n} = \frac{1}{3} \cdot \left(\frac{3}{2}\right)^n$

GEOM SERIES $r > 1$ DIVERGES!

$u_k = 2^{-k} \cdot 3^{k-1} \quad \left(\frac{3^{k-1}}{2^k}\right)$

$u_{k+1} = 2^{-k-1} \cdot 3^{k} \quad \left(\frac{3^k}{2^{k+1}}\right)$

$\lim_{k\to\infty} \frac{2^{-k-1} \cdot 3^k}{2^{-k} \cdot 3^{k-1}} = 2^{-1} \cdot 3 = \frac{3}{2} > 1$ DIVERGES BY RATIO TEST.

33.) $\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^{n+1}}{\sqrt{n}}$

$u_k = \dfrac{(-5)^{k+1}}{\sqrt{k}}$ $\quad |u_k| = \dfrac{5^{k+1}}{\sqrt{k}}$

$|u_{k+1}| = \dfrac{5^{k+2}}{\sqrt{k+1}}$

$\lim\limits_{k\to\infty} \dfrac{5^{k+2}\cdot\sqrt{k}}{5^{k+1}\cdot\sqrt{k+1}} = 5\cdot\sqrt{\dfrac{k}{k+1}} = 5 > 1$ DIVERGES

34.) $\displaystyle\sum_{n=1}^{\infty} e^{-n}\cdot n!$

$|u_k| = e^{-k}\cdot k! = \dfrac{k!}{e^k}$

$|u_{k+1}| = e^{-k-1}(k+1)! = \dfrac{(k+1)!}{e^{k+1}}$

$\lim\limits_{k\to\infty} \dfrac{(k+1)!\;e^k}{k!\cdot e^{k+1}} = (k+1)\cdot\dfrac{1}{e} = \infty > 1$ DIVERGES

35.) $\sum_{n=3}^{\infty} \frac{n^2+1}{n^3+1}$ $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n^2+1}{n^3+1}} = \frac{n^3+1}{n(n^2+1)} = \frac{n^3+1}{n^3+n} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \frac{1+\frac{1}{n^3}}{1+\frac{1}{n^2}} = 1 > 0$$

WORK TOGETHER

$\sum_{n=3}^{\infty} \frac{1}{n}$ DIVERGES BY P-SERIES TEST, $\therefore \sum_{n=3}^{\infty} \frac{n^2+1}{n^3+1}$ DIVERGES BY THE LIMIT COMPARISON TEST.

36.) Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2n}{4n^2-3}$ conditionally converges, absolutely converges, or diverges.

ALT SERIES u_n CONV., BUT $|u_n|$ DOES NOT.

$$(u_n) = \frac{2k}{4k^2-3} > \frac{2k}{4k^2} = \frac{1}{2k}$$

$$\frac{1}{2k} < \frac{2k}{4k^2-3}$$

$\sum_{n=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{k}$ DIVERGES BY P-SERIES TEST.

$\therefore \sum_{n=1}^{\infty} \frac{2n}{4n^2-3}$ DIVERGES BY COMPARISON TEST.

ALT. SERIES TEST

- $\lim_{x\to\infty} = 0$ (BOTTOM HEAVY) ✓
- $|u_{n+1}| < |u_n|$ ✓

$n=1$ $|u_1| = 2$

$n=2$ $|u_2| = 4/13$ 0.308

$n=3$ $|u_3| = 2/11$ 0.182

$n=4$ $|u_4| = 8/61$ 0.131

37.) Determine the interval of convergence for the series $\sum_{k=1}^{\infty}\frac{(x-2)^k}{k\cdot 5^k}$.

$|u_k| = \frac{(x-2)^k}{k\cdot 5^k}$ $\quad |u_{k+1}| = \frac{(x-2)^{k+1}}{(k+1)\,5^{k+1}}$

RATIO TEST

$$\lim_{k\to\infty} \frac{(x-2)^{k+1}\cdot k\cdot 5^k}{(x-2)^k\cdot(k+1)\cdot 5^{k+1}} = (x-2)\cdot\left(\frac{\cancel{k}}{\cancel{k+1}}\right)^{1}\cdot\frac{1}{5} = \frac{1}{5}(x-2)$$

$-1 < \frac{1}{5}(x-2) < 1$

$-5 < x-2 < 5$

$-3 < x < 7$

AT $x=-3$

$\frac{-5^k}{k\cdot 5^k} = (-1)^k\frac{1}{k}$

ALT. SERIES

- $\lim_{k\to\infty} = 0$ ✓
- $|u_{k+1}| < |u_k|$ ✓

CONVERGES (ALT. SERIES TEST)

AT $x=7$

$\frac{5^k}{k\cdot 5^k} = \frac{1}{k}$

DIVERGES (P-SERIES TEST)

INTERVAL OF CONVERGENCE

$\boxed{-3 \le x < 7}$

38.) Find the Maclaurin series expansion of the function below up to the x^3 term

$$f(x) = \cos\left(\frac{x}{2}\right)\cdot e^{-x}$$

$e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

$\cos\left(\frac{x}{2}\right) = 1 - \frac{\left(\frac{x}{2}\right)^2}{2!} + \frac{\left(\frac{x}{2}\right)^4}{4!}$

$= 1 - \frac{x^2}{8} + \frac{x^4}{384}$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$

$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

$\left(1-\frac{x^2}{8}\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^2}{8} + \frac{x^3}{8} - \cancel{\frac{x^4}{16}} + \cancel{\frac{x^5}{48}}$

$= \boxed{1 - x + \frac{3}{8}x^2 - \frac{1}{24}x^3}$

39.) Find the Maclaurin series expansion of the function below up to the x^3 term

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$g(x) = \arctan(\sin 4x)$$

$\sin 4x = 4x - \frac{(4x)^3}{3!} = 4x - \frac{32}{3}x^3$

$\arctan\left(4x - \frac{32}{3}x^3\right) = \left(4x - \frac{32}{3}x^3\right) - \frac{\left(4x - \frac{32}{3}x^3\right)^3}{3} + \ldots$

$= 4x - \frac{32}{3}x^3 - \frac{1}{3}\left(64x^3 - \cancel{3(16x^2)\left(\frac{32}{3}x^3\right)} + \cancel{3(4x)\left(\frac{1024}{9}x^6\right)} - \cancel{\frac{32768}{27}x^9}\right)$

BINOMIAL THEOREM

OTHER TERMS > x^3 !!

$= 4x - \frac{32}{3}x^3 - \frac{64}{3}x^3$

$= \boxed{4x - \frac{96}{3}x^3} = \boxed{4x - 32x^3}$

40.) Find the first three terms of the Maclaurin series for $\ln(1+e^x)$.

$\ln\left(1 + \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3}\right)\right) = \left(1 + x + \frac{x^2}{2}\right) - \sim$ WON'T WORK!

$f'(x) = \frac{1}{1+e^x} \cdot e^x = \frac{e^x}{1+e^x}$

$f''(x) = \frac{(1+e^x)(e^x) - (e^x)(e^x)}{(1+e^x)^2} = \frac{e^x(1+e^x-e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$

SKIP $f'''(x) = \cancel{\frac{(1+e^x)^2(e^x) - (e^x)(2)(1+e^x)(e^x)}{(1+e^x)^4}} = \cancel{(1+e^x)}e^x$ ONLY NEED 3 TERMS

$f(0) = \ln 2$

$f'(0) = \frac{1}{2}$

$f''(0) = \frac{1}{4}$

$f(x) = \ln 2 + \frac{\frac{1}{2}x}{1!} + \frac{\frac{1}{4}x^2}{2!}$

$\boxed{f(x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2}$

41.) Given that $y = \ln(\cos x)$, CAN'T USE $\ln(1+x)$ GO 4 DERIV. DEEP!

a. Show that the first two non-zero terms of the Maclaurin series for y are $-\frac{x^2}{2} - \frac{x^4}{12}$.

$f'(x) = \frac{1}{\cos x} \cdot -\sin x = -\tan x$

$f''(x) = -\sec^2 x = -(\sec x)^2$

$f'''(x) = -2\sec x \sec x \tan x = -2(\sec x)^2 \tan x$

$f^{(4)}(x) = -2(2)(\sec x \sec x \tan x \tan x) + (-2(\sec x)^2 \sec^2 x) = \boxed{-4\sec^2 x \tan^2 x - 2\sec^4 x}$

$f(0) = 0$

$f'(0) = 0$

$f''(0) = -1$

$f'''(0) = 0$

$f^{(4)}(0) = -2$

$f(x) = 0 + 0 + \frac{-1}{2!}x^2 + 0 + \frac{-2}{4!}x^4 = \boxed{-\frac{1}{2}x^2 - \frac{1}{12}x^4}$

b. Use this series to find an approximation in terms of π for $\ln 2$.

$\ln 2 = \ln\left(\frac{1}{2}\right)^{-1} = -\ln\left(\frac{1}{2}\right)$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$

$= -\left(-\frac{1}{2}\left(\frac{\pi}{3}\right)^2 - \frac{1}{12}\left(\frac{\pi}{3}\right)^4\right)$

$= \frac{1}{2} \cdot \frac{\pi^2}{9} + \frac{1}{12} \cdot \frac{\pi^4}{81}$

$= \boxed{\frac{\pi^2}{18} + \frac{\pi^4}{972}}$

42.) Find the Maclaurin series for $f(x)=\frac{1}{1+x^2}$ as far as the x^3 term.

$$\boxed{\begin{array}{l} f'(x)=\frac{1}{1+x^2} \\ f(x)=\arctan x \end{array}}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$$\frac{1}{1+x^2} = 1 - \frac{3x^2}{3} + \frac{5x^4}{5} - \dots$$

$$= \underline{1 - x^2} + x^4 - \dots$$

$$\frac{1}{1+x^2} = \boxed{1 - x^2}$$

43.) Use a Maclaurin series to approximate the integral, $\int_0^1 e^{x^3}\,dx$.

$$e^{x^3} = 1 + (x^3) + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \dots$$

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \dots$$

$$\int_0^1 e^{x^3} \cdot dx = \left[x + \tfrac{1}{4}x^4 + \tfrac{1}{14}x^7 + \tfrac{1}{60}x^{10} + \dots\right]_0^1$$

$$\underline{1 + \frac{1}{4} + \frac{1}{14} + \frac{1}{60} + \frac{1}{312} + \frac{1}{1920}}$$

$$= \left(1 + \frac{1}{4} + \frac{1}{14} + \frac{1}{60}\right) - (0)$$

$$= \boxed{1.338}$$ (ACTUAL: $\underline{1.34190441798}$)

44.) Find the first 3 non-zero terms of the Maclaurin series for $x^2 \cdot \cos 2x$.

$$f(x) = x^2 \cdot \cos(2x)$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}$$

$$= 1 - \frac{4x^2}{2} + \frac{16x^4}{24}$$

$$\boxed{= 1 - 2x^2 + \tfrac{2}{3}x^4}$$

$$\boxed{x^2 \cdot \cos(2x) = x^2 - 2x^4 + \tfrac{2}{3}x^6}$$

45.) Find the Taylor polynomial of degree 2 for $f(x)=\dfrac{1}{x+2}$ centered about $x=3$.

$f(x)=\frac{1}{x+2}$ $\quad f(3)=\frac{1}{5}$

$f'(x)=\frac{-1}{(x+2)^2}$ $\quad f'(3)=-\frac{1}{25}$

$f''(x)=\frac{2}{(x+2)^3}$ $\quad f''(3)=\frac{2}{125}$

$f(x)=\frac{1/5}{\cancel{0!}}\cancel{(x-3)^0}+\frac{-1/25}{1!}(x-3)^1+\frac{2/125}{2!}(x-3)^2$

$\boxed{f(x)=\frac{1}{5}-\frac{1}{25}(x-3)+\frac{1}{125}(x-3)^2}$

46.) Find the Maclaurin series for $\ln(1+\sin x)$ up to and including the x^2 term.

$\ln\left(1+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)\right) = \sim$ WON'T WORK !

$f(0)=0$

$f'(0)=1$

$f''(0)=-1$

$f(x)=\boxed{\ln(1+\sin x)}$

$f'(x)=\frac{1}{1+\sin x}\cdot\cos x=\boxed{\frac{\cos x}{1+\sin x}}$

$f''(x)=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}=\frac{-\sin x-1}{(1+\sin x)^2}=\boxed{\frac{-1}{1+\sin x}}$

$f(x)=\frac{0}{\cancel{0!}}\cancel{x^0}+\frac{1}{1!}x^1+\frac{-1}{2!}x^2=\boxed{x-\frac{1}{2}x^2}$

47.) Without using the any of the shortened formulas, find the Maclaurin series for the function $f(x)=xe^x$ up to the x^4 term. Express your final answer as a series in the form $\sum a_n\cdot x^n$.

Note: You must show the correct calculations including all necessary derivatives.

$f(x)=x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)$
$=x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4$

$f(x)=xe^x$ $\quad f(0)=0$

$f'(x)=\boxed{1\cdot e^x+xe^x}$ $\quad f'(0)=1$

$f''(x)=e^x+1\cdot e^x+xe^x=\boxed{2e^x+xe^x}$ $\quad f''(0)=2$

$f'''(x)=2e^x+1\cdot e^x+xe^x=\boxed{3e^x+xe^x}$ $\quad f'''(0)=3$

$f^{(4)}(x)=\boxed{4e^x+xe^x}$ $\quad f^{(4)}(0)=4$

$f^{(n)}(x)=ne^x+xe^x$

$f(x)=\frac{0}{\cancel{0!}}\cancel{x^0}+\frac{1}{1!}x^1+\frac{2}{2!}x^2+\frac{3}{3!}x^3+\frac{4}{4!}x^4$

$f(x)=\frac{x^1}{0!}+\frac{x^2}{1!}+\frac{x^3}{2!}+\frac{x^4}{3!}+\ldots$

$\boxed{f(x)=\sum_{k=1}^{\infty}\frac{x^{k+1}}{k}}$

48.) Find the first three nonzero terms in the Maclaurin series for $f(x)=e^{x}\sin(-2x)$.

$$e^{x}\cdot\sin(-2x)=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left((-2x)-\frac{(-2x)^3}{3!}+\frac{(-2x)^5}{5!}\right)$$

$$=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)\left(-2x+\frac{4}{3}x^3-\frac{4}{15}x^5\right)$$

$$=\underline{-2x+\frac{4}{3}x^3-2x^2+\cancel{\frac{4}{3}x^4}-x^3}$$ (SKIPPED ALL TERMS $\geq x^4$)

$$=\boxed{-2x-2x^2+\frac{1}{3}x^3}$$

49.) Find the 3rd degree Taylor series for the function $f(x)=e^{-x}$ centered about $x=\ln 2$.

$f(x)=e^{-x}$ $\quad f(\ln 2)=\frac{1}{2}$

$f'(x)=-e^{-x}$ $\quad f'(\ln 2)=-\frac{1}{2}$

$f''(x)=e^{-x}$ $\quad f''(\ln 2)=\frac{1}{2}$

$f'''(x)=-e^{-x}$ $\quad f'''(\ln 2)=-\frac{1}{2}$

$$f(x)=\frac{\frac{1}{2}}{0!}\cancel{(x-\ln 2)^0}+\frac{-\frac{1}{2}}{1!}(x-\ln 2)^1+\frac{\frac{1}{2}}{2!}(x-\ln 2)^2+\frac{-\frac{1}{2}}{3!}(x-\ln 2)^3$$

$$\boxed{=\frac{1}{2}-\frac{1}{2}(x-\ln 2)+\frac{1}{4}(x-\ln 2)^2-\frac{1}{12}(x-\ln 2)^3}$$

50.) Use the first 4 terms of a Maclaurin series to approximate the integral, $\int_0^1 \sin\left(x^2\right)dx$. You may express your answer as a decimal or a fraction in reduced form.

$$\sin x^2=(x^2)-\frac{(x^2)^3}{3!}+\frac{(x^2)^5}{5!}-\frac{(x^2)^7}{7!}$$

$$=x^2-\frac{1}{6}x^6+\frac{1}{120}x^{10}-\frac{1}{5040}x^{14}$$

$$\int_0^1\left(x^2-\frac{1}{6}x^6+\frac{1}{120}x^{10}-\frac{1}{5040}x^{14}\right)dx$$

$$\left[\frac{1}{3}x^3-\frac{1}{42}x^7+\frac{1}{1320}x^{11}-\frac{1}{75600}x^{15}\right]_0^1$$

$$\left(\frac{1}{3}-\frac{1}{42}+\frac{1}{1320}-\frac{1}{75600}\right)-(0)$$

$\underline{0.310}$

ACTUAL:

$\underline{0.310268301723}$