1\) a.) $f(x) = \frac{e^x + e^{-x}}{2}$

$f'(x) = \frac{1}{2}(e^x + e^{-x} \cdot -1)$

$f'(x) = \frac{1}{2}(e^x - e^{-x})$

$f'(x) = g(x)$ ✓

$g(x) = \frac{e^x - e^{-x}}{2}$

$g'(x) = \frac{1}{2}(e^x - e^{-x} \cdot (-1))$

$g'(x) = \frac{1}{2}(e^x + e^{-x})$

$g'(x) = f(x)$ ✓

$\underline{f(0) = 1}$

$\underline{g(0) = 0}$

b.) $f(0) = 1$

$f'(0) = g(0) = 0$

$f''(0) = f(0) = 1$

$f'''(0) = 0$

$f^{(4)}(0) = 1$

$f(x) = \frac{1}{0!}x^0 + \frac{0}{1!}x^1 + \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4$

$\boxed{f(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^2}$

c.) $\lim_{x \to 0} \frac{1 - f(x)}{x^2} \left(\frac{0}{0}\right) = \frac{-f'(x)}{2x} = \frac{-g(x)}{2x} \left(\frac{0}{0}\right) = \frac{-g'(x)}{2} = \frac{-f(x)}{2} = \boxed{-\frac{1}{2}}$

L'HOPITAL'S RULE TWICE

d.) $\int_0^\infty \frac{g(x)}{[f(x)]^2} \cdot dx = \lim_{b \to \infty} \int_0^b \frac{g(x)}{[f(x)]^2} \cdot dx = \int_0^b \frac{1}{u^2}\, du = \left[-\frac{1}{u}\right]_0^b = \left[-\frac{1}{f(x)}\right]_0^b = \underline{-\frac{1}{f(b)} + 1}$

$u = f(x)$

$du = f'(x)\, dx = g(x)\, dx$

$\lim_{b \to \infty} -\frac{1}{f(b)} + 1 = \left(\lim_{x \to \infty} \frac{e^x + e^{-x}}{2} = \frac{1}{2}e^x + \frac{1}{2}\frac{1}{e^x} = \infty \right) = -\frac{1}{\infty} + 1 = \boxed{1}$

2\) i. $f(x) = (\ln x)^2$

$f'(x) = 2(\ln x) \cdot \frac{1}{x} = \boxed{\frac{2 \ln x}{x}}$

ii. $g(x) = \ln(f(x))$

$g'(x) = \frac{1}{f(x)} \cdot f'(x) = \frac{2 \ln x}{x} \cdot \frac{1}{(\ln x)^2} = \boxed{\frac{2}{x \cdot \ln x}}$

iii. $g'(x) = 0$

$\frac{2}{x \ln x} = 0 \quad 2 = 0$ NO SOLUTION, THUS NO MAX/MIN. FOR $x > 1$, $\frac{2}{x \ln x} > 0$ $\therefore$ IT IS ALWAYS INC.

SEE OTHER SHEET.

$I(x) = e^{\int \frac{2}{x \cdot \ln x} \cdot dx} = e^{\ln((\ln x)^2)} = \underline{(\ln x)^2}$

2 b.) Consider the differential equation

$$(\ln x)\frac{dy}{dx}+\frac{2}{x}y=\frac{2x-1}{(\ln x)},\ x>1.$$

(i) Find the general solution of the differential equation in the form $y=h(x)$.

(ii) Show that the particular solution passing through the point with coordinates (e, e^2) is given by $y=\dfrac{x^2-x+e}{(\ln x)^2}$.

(iii) Sketch the graph of your solution for $x>1$, clearly indicating any asymptotes and any maximum or minimum points. *[12]*

(i.) DIVIDE OUT $(\ln x)$

$$\frac{dy}{dx}+\frac{2}{x\cdot\ln x}\cdot y=\frac{2x-1}{(\ln x)^2}$$

$$I(x)=e^{\int\frac{2}{x\cdot\ln x}\cdot dx}$$

$$\int\frac{2}{x\cdot\ln x}\cdot dx \qquad u=\ln x \qquad du=\frac{1}{x}\cdot dx$$

$$\int\frac{2}{u}\cdot du=2\ln|u|=\ln u^2=\ln(\ln x)^2$$

$$I(x)=e^{\ln(\ln x)^2}=(\ln x)^2$$

MULTIPLY BY $I(x)$

$$(\ln x)^2\cdot\frac{dy}{dx}+\frac{2}{x}\cdot\ln x\cdot y=2x-1$$

$$y(\ln x)^2=x^2-x+C$$

$$\boxed{y=\frac{x^2-x+C}{(\ln x)^2}}$$

(ii.)

$$e^2=\frac{e^2-e+C}{(\ln e)^2}$$

$$e^2=e^2-e+C$$

$$C=e$$

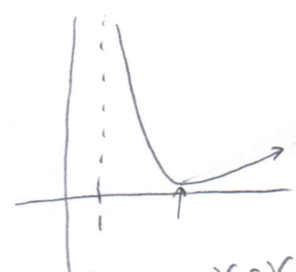
$$\boxed{y=\frac{x^2-x+e}{(\ln x)^2}}$$

(iii) USE GRAPHING CALCULATOR

V.A. AT

$x=1$

(NOT ON $x>1$)



$$\frac{dy}{dx}=\frac{(\ln x)^2(2x-1)-(x^2-x+e)(2)(\ln x)(\frac{1}{x})}{(\ln x)^4}$$

$$\frac{dy}{dx}=0 \qquad \ln x\left((\ln x)(2x-1)-\left(2x-2+\frac{2e}{x}\right)\right)=0$$

USE G.D.C MIN AT $x=3.128$

③ a) $b(n) = 1, 4, 7, 10 \quad 3n+1 \qquad b(n) = 3n+1$

$c(n) = 2, 5, 8, 11 \quad 3n+2 \qquad c(n) = 3n+2$

$(0)\ (1)\ (2)\ (3)$

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)} x^n$$

b.) RATIO TEST

$$|u_k| = \frac{1}{(3k+1)(3k+2)} x^k \qquad |u_{k+1}| = \frac{1}{(3k+2)(3k+3)} x^{k+1}$$

$$\lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \frac{x^{k+1}}{x^k} \cdot \frac{(3k+1)(3k+2)}{(3k+2)(3k+3)} = x$$

RADIUS OF CONV. $\boxed{= 1}$

$\underline{-1 < x < 1}$

c.) AT $x=-1$

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(3n+1)(3n+2)} \qquad \text{ALT. SERIES}$$

- $\lim_{n\to\infty} = 0$ ✓
- $|u_{k+1}| < |u_k|$ ✓

CONV.

AT $x = 1$

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)} \qquad \underline{9n^2+9n+2}$$

LIMIT COMP. TEST

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{9n^2+9n+2}} = \frac{9n^2+9n+2}{n^2} = 9 + \frac{9}{n} + \frac{2}{n^2} = 9 > 0$$

WORK TOGETHER

$\frac{1}{n^2}$ CONV. BY P-SERIES TEST, $\therefore \frac{1}{(3n+1)(3n+2)}$ CONV. BY LIMIT. COMP. TEST

INTERVAL CONV. $\boxed{-1 \le x \le 1}$

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④ $$f(x) = \begin{cases} e^{-x^2}(-x^3+2x^2+x) & x \le 1 \\ ax+b & x > 1 \end{cases}$$

b.) ON NEXT PAGE.

a.) TOP $f(1) = \frac{1}{e}(-1+2+1) = \frac{2}{e}$

BOTTOM $f(1) = a+b$

$a+b = \frac{2}{e}$

$$f'(x) = \begin{cases} (-2x \cdot e^{-x^2})(-x^3+2x^2+x) + e^{-x^2}(-3x^2+4x+1) \\ a \end{cases}$$

$$= e^{-x^2}(2x^4 - 4x^3 - 2x^2 - 3x^2 + 4x + 1)$$

$$= e^{-x^2}(2x^4 - 4x^3 - 5x^2 + 4x + 1)$$

USED IN PART 2

$$f'(1) = \begin{cases} -\frac{2}{e}(-1+2+1) + \frac{1}{e}(-3+4+1) \\ a \end{cases}$$

$$= \begin{cases} -\frac{4}{e} + \frac{2}{e} \\ a \end{cases}$$

$$= \begin{cases} -\frac{2}{e} \\ a \end{cases} \qquad \underline{a = -\frac{2}{e}} \qquad \underline{b = \frac{4}{e}}$$

4. b.) $\boxed{i.}$ $2x^4 - 4x^3 - 5x^2 + 4x + 1 = 0$

ROLLE'S THM: IF $f(a) = f(b)$, THEN $c \in]a,b[$ SUCH THAT $f'(c) = 0$.

$f(1) = \frac{2}{e}$ (FROM PART (a))

$f(-1) = \frac{1}{e}(1 + 2 - 1) = \frac{2}{e}$

$f(1) = f(-1)$ ✓

$f'(x) = \underline{e^{-x^2}(2x^4 - 4x^3 - 5x^2 + 4x + 1)} = 0$

THERE MUST EXIST $c \in]-1,1[$ WHERE $f'(c) = 0$.

SINCE $e^{-x^2} = \frac{1}{e^{x^2}}$ CANNOT $= 0$, $2x^4 - 4x^3 - 5x^2 + 4x + 1 = 0$ AT LEAST ONCE ON $]-1,1[$.

$\boxed{ii.}$ LET $g(x) = 2x^4 - 4x^3 - 5x^2 + 4x + 1$

$g(x)$ IS A POLYNOMIAL FXN SO IS CONTINUOUS.

$g(-1) = 2 + 4 - 5 - 4 + 1 = -2$

$g(0) = 1$

$g(1) = 2 - 4 - 5 + 4 + 1 = -2$

$\therefore$ $g(x)$ MUST HAVE AT LEAST 1 ROOT ON $[-1,0]$ AND 1 ON $[0,1]$.