68. The polynomial $f(x) = x^2 + 9x + 33$ has roots of a and b. Without finding the actual roots, find the exact value of the expression $\frac{1}{a^2} + \frac{1}{b^2}$ using Viete's Theorem. Leave your answer as a reduced fraction.

VIETE'S THM

$a + b = -\frac{b}{a} = \frac{-9}{1} = -9$

$a \cdot b = \frac{c}{a} = \frac{33}{1} = 33$

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{b^2 + a^2}{a^2 b^2}$$

$$= \frac{a^2 + 2ab + b^2 - 2ab}{(ab)^2}$$

$$= \frac{(a+b)^2 - 2(ab)}{(ab)^2}$$

$$= \frac{(-9)^2 - 2(33)}{(33)^2}$$

$$= \frac{15}{1089}$$

$$\boxed{= \frac{5}{363}}$$

(Total 7 marks)

---

69. (a) Show that the complex number i is a root of the equation
$$x^4 - 5x^3 + 7x^2 - 5x + 6 = 0.$$
(2)

$$\begin{array}{c|ccccc} i & 1 & -5 & 7 & -5 & 6 \\ & \downarrow & i & -1-5i & 5+6i & -6 \\ \hline & 1 & -5+i & 6-5i & 6i & \boxed{0} \end{array}$$ ✓

(b) Find the other roots of this equation.
(4)

CONJ. ROOT THM

$$\begin{array}{c|cccc} -i & 1 & -5+i & 6-5i & 6i \\ & \downarrow & -i & 5i & -6i \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$(x^2 - 5x + 6) = (x-3)(x-2)$

$\boxed{\text{ROOTS: } \pm i,\ 2,\ 3}$

(Total 6 marks)

---

70. Given that $\frac{z}{z+2} = 2 - \text{i}$, $z \in \mathbb{C}$, find z in the form $a + \text{i}b$.

$z = (2-i)(z+2)$

$z = 2z + 4 - iz - 2i$

$-1z + iz = 4 - 2i$

$z(-1+i) = 4 - 2i$

$$z = \frac{4-2i}{-1+i} \cdot \frac{(-1-i)}{(-1-i)}$$

$$z = \frac{-4 - 4i + 2i + 2i^2}{1 - i^2}$$

$$z = \frac{-6-2i}{2}$$

$$\boxed{z = -3 - i}$$

(Total 4 marks)

---

71. When $f(x) = x^4 + 3x^3 + px^2 - 2x + q$ is divided by $(x - 2)$ the remainder is 15, and $(x + 3)$ is a factor of $f(x)$. Find the values of p and q.

$$\begin{array}{c|ccccc} 2 & 1 & 3 & p & -2 & q \\ & \downarrow & 2 & 10 & 2p+20 & 4p+36 \\ \hline & 1 & 5 & p+10 & 2p+18 & \boxed{15} \end{array}$$

$q + 4p + 36 = 15$

$4p + q = -21$

$$\begin{array}{c|ccccc} -3 & 1 & 3 & p & -2 & q \\ & \downarrow & -3 & 0 & -3p & 9p+6 \\ \hline & 1 & 0 & p & -3p-2 & \boxed{0} \end{array}$$

$q + 9p + 6 = 0$

$9p + q = -6$

REMAINDER = 0

$4p + q = -21$

$-9p - q = 6$

$-5p = -15$

$\boxed{p = 3 \quad q = -33}$

(Total 6 marks)

72. Given that $(a + i)(2 - bi) = 7 - i$, find the value of a and b, where $\boxed{a, b \in \mathbb{Z}}$.

$2a - abi + 2i - bi^2 = 7 - i$

$(2a + b) + (-ab + 2)i = 7 - i$

$2a + b = 7 \qquad b = -2a + 7$

$-ab + 2 = -1 \qquad -a(-2a + 7) + 2 = -1$

$2a^2 - 7a + 3 = 0$

$(2a - 1)(a - 3) = 0$

$a = \cancel{\tfrac{1}{2}}, 3$

$\boxed{\begin{array}{l} a = 3 \\ b = 1 \end{array}}$

(Total 6 marks)

73. (a) Evaluate $(1 + i)^2$, where $i = \sqrt{-1}$. **(2)**

$1 + 2i + i^2 = 1 + 2i - 1 = \boxed{2i}$

(b) Show that $(1 + i)^{4n} = (-4)^n$, where $n \in \mathbb{N}$. **(4)**

$((1 + i)^4)^n = (((1+i)^2)^2)^n = ((2i)^2)^n = (4i^2)^n = \boxed{(-4)^n}$ ✓

(c) Hence or otherwise, find $(1 + i)^{32}$. **(2)**

$(1+i)^{32} = (1+i)^{4 \cdot 8} = (-4)^8 = \boxed{65{,}536}$

(Total 8 marks)

74. The cubic equation $f(x) = 3x^3 + x^2 - 6x + 20$ has roots of x, y and z. Without finding the actual roots, find the exact value of the expression $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ using Viete's Theorem. Leave your answer as a reduced fraction.

$x + y + z = -\frac{b}{a} = -\frac{1}{3}$

$xy + xz + yz = \frac{c}{a} = -2$

$xyz = -\frac{d}{a} = -\frac{20}{3}$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{yz + xz + xy}{xyz}$

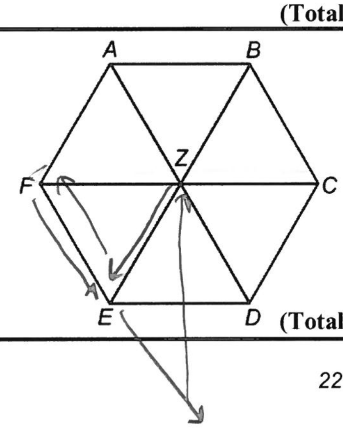
$= \frac{-2}{-\frac{20}{3}}$

$= -2 \cdot -\frac{3}{20}$

$\boxed{= \frac{3}{10}}$

(Total 6 marks)

75. Use the regular hexagon to the right to answer parts (a) and (b).



a.) List **all** equivalent vectors to $-\frac{1}{2}\overrightarrow{CF}$. $= \frac{1}{2}\overrightarrow{FC}$

$\boxed{\overrightarrow{FZ},\ \overrightarrow{ZC},\ \overrightarrow{AB},\ \overrightarrow{ED}}$

b.) Write an equivalent vector to $2\overrightarrow{FE} + \overrightarrow{DB} - \overrightarrow{ZB} + \overrightarrow{ZA}$. $\boxed{= 0}$

$+\overrightarrow{BZ}$

NULL VECTOR

(Total 5 marks)

76. The three vectors ***a***, ***b*** and ***c*** are given by

$$a = \begin{pmatrix} 2y \\ -3x \end{pmatrix}, b = \begin{pmatrix} 4x \\ y \end{pmatrix}, c = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \text{ where } x, y \in \mathbb{R}.$$

$4x + y = 2$

$y = -4x + 2$

(a) If ***a*** + 2***b*** – ***c*** = 0, find the value of x and of y. **(3)**

$$\begin{pmatrix} 2y \\ -3x \end{pmatrix} + \begin{pmatrix} 8x \\ 2y \end{pmatrix} + \begin{pmatrix} -4 \\ 7 \end{pmatrix} = 0$$

$2y + 8x - 4 = 0$

$-3x + 2y + 7 = 0$

$8x + 2y = 4$

$-3x + 2y = -7$

$8x + \cancel{2y} = 4$

$3x - \cancel{2y} = 7$

$11x = 11$

$\boxed{x = 1 \quad y = -2}$

(b) Find the exact value of $|\boldsymbol{a} + 2\boldsymbol{b}|$. **(2)**

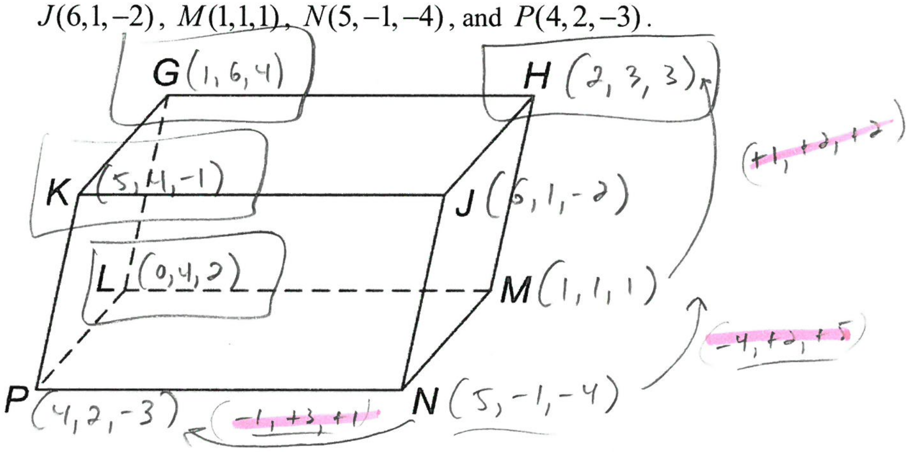
$$a = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad 2b = \begin{pmatrix} 8 \\ -4 \end{pmatrix} \quad a + 2b = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \quad |a + 2b| = \sqrt{16 + 49} = \boxed{\sqrt{63}}$$

(Total 5 marks)

---

77. The parallelepiped below (all 6 sides are parallelograms) has the following vertices:

$J(6,1,-2)$, $M(1,1,1)$, $N(5,-1,-4)$, and $P(4,2,-3)$.



a.) Find the coordinates of the remaining 4 points.

SEE ABOVE

b.) Find $\overrightarrow{PH}$.

$$\overrightarrow{PH} = \begin{pmatrix} 2-4 \\ 3-2 \\ 3-(-3) \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}}$$

c.) Find $2\overrightarrow{MN} \boxed{-\overrightarrow{JG}}$

$-\overrightarrow{JG} = \overrightarrow{GJ}$

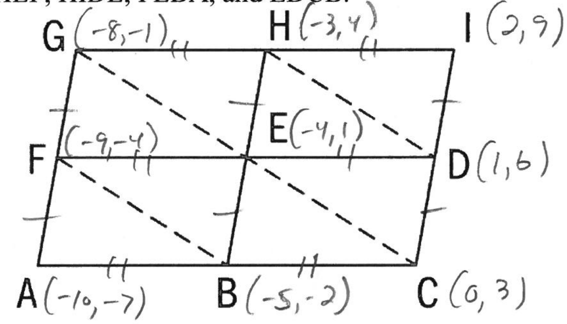
$$\overrightarrow{MN} = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix}$$

$$2\overrightarrow{MN} = \begin{pmatrix} 8 \\ -4 \\ -10 \end{pmatrix} \quad \overrightarrow{GJ} = \begin{pmatrix} 5 \\ -5 \\ -6 \end{pmatrix}$$

$$2\overrightarrow{MN} + \overrightarrow{GJ} = \begin{pmatrix} 8+5 \\ -4-5 \\ -10-6 \end{pmatrix} = \boxed{\begin{pmatrix} 13 \\ -9 \\ -16 \end{pmatrix}}$$

(Total 8 marks)

78. The diagram below is made up of four identical parallelograms GHEF, HIDE, FEBA, and EDCB. Let $G(-8,-1)$, $H(-3,4)$, and $B(-5,-2)$.



a.) Find $\overrightarrow{AI}$.

$$\overrightarrow{AI} = \begin{pmatrix} 2-(-10) \\ 9-(-7) \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

b.) Find the magnitude of $\overrightarrow{CF}$.

$$\overrightarrow{CF} = \begin{pmatrix} -9 \\ -7 \end{pmatrix} \quad |\overrightarrow{CF}| = \sqrt{81+49} = \boxed{\sqrt{130}}$$

c.) Find the unit vectors collinear to $\overrightarrow{DH}$.

$$\overrightarrow{DH} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \quad |\overrightarrow{DH}| = \sqrt{16+4} = 2\sqrt{5} \quad u = \pm \frac{1}{2\sqrt{5}}\begin{pmatrix} -4 \\ -2 \end{pmatrix} = \boxed{\pm \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ -1 \end{pmatrix}}$$

d.) Find a vector with the same magnitude as $\overrightarrow{AH}$ in the same direction as $\overrightarrow{AG}$. Write your answer in the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where $x, y \in \mathbb{R}$.

$$\overrightarrow{AG} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad v = \frac{\sqrt{170}}{\sqrt{40}}\begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$\overrightarrow{AH} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}. \quad |\overrightarrow{AH}| = \sqrt{49+121} = \sqrt{170} \quad |\overrightarrow{AG}| = \sqrt{4+36} = \sqrt{40} \quad v = \frac{\sqrt{17}}{2}\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \boxed{\sqrt{17}\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

(Total 13 marks)

79. Given the points $P(6,0,3)$, $Q(2,-4,-1)$, and $R(-1,7,-3)$, find:

(a) A vector equation of line $\overleftrightarrow{PQ}$.

$$\overrightarrow{PQ} = \begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix} \quad \boxed{l_1 = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} + a\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}} \leftarrow \text{SCALED DOWN BY } 1/4$$

(b) Parametric equations for line $\overleftrightarrow{QR}$.

$$\overrightarrow{QR} = \begin{pmatrix} -3 \\ 11 \\ -2 \end{pmatrix} \quad l_2 = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} + \beta\begin{pmatrix} -3 \\ 11 \\ -2 \end{pmatrix}$$

$$\boxed{\begin{aligned} x &= 2 - 3\beta \\ y &= -4 + 11\beta \\ z &= -1 - 2\beta \end{aligned}}$$

(c) Cartesian equations for line $\overleftrightarrow{PR}$.

$$\overrightarrow{PR} = \begin{pmatrix} -7 \\ 7 \\ -6 \end{pmatrix} \quad l_3 = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -7 \\ 7 \\ -6 \end{pmatrix} \quad \left.\begin{aligned} x &= 6 - 7\lambda \\ y &= 0 + 7\lambda \\ z &= 3 - 6\lambda \end{aligned}\right\} \text{SOLVE FOR } \lambda$$

$$\boxed{\frac{x-6}{-7} = \frac{y}{7} = \frac{z-3}{-6}}$$

(d) Any 3 points **other than P, Q, and R** which lie on line $\overleftrightarrow{QR}$.

FROM PART (b)

β	-3	-2	-1	0	1	2	3	4
POINT	(11,-37,5)	(8,-26,3)	(5,-15,1)	(2,-4,-1)	(-1,7,-3)	(-4,18,-5)	(-7,29,-7)	(-10,40,-9)
				Q	R			

(e) Find the value of $\overrightarrow{PQ} \bullet \overrightarrow{PR}$.

$$\begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 7 \\ -6 \end{pmatrix} = 28 - 28 + 24 = \boxed{24}$$

(Total 11 marks)

80. A triangle has its vertices at A(–1, 3, 2), B(3, 6, 1) and C(–4, 4, 3).

(a) Find $\overrightarrow{AB} \bullet \overrightarrow{AC}$.

$$\overrightarrow{AB}\cdot\overrightarrow{AC} = \begin{pmatrix}4\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\1\end{pmatrix} = -12+3-1 = \boxed{-10}$$

(b) To one decimal place, find the measure of $\angle BAC$.

$$\angle BAC = \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}\right) = \cos^{-1}\left(\frac{-10}{\sqrt{26}\sqrt{11}}\right) = \boxed{126.3^\circ}$$

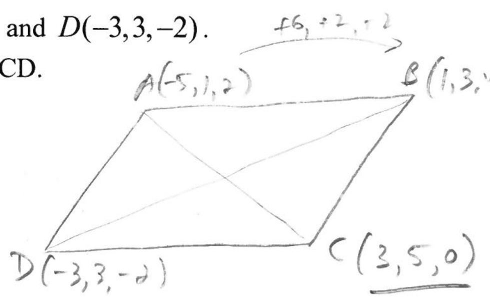
(c) Is $\angle CBA$ acute or obtuse? Briefly explain.

$\boxed{\text{ACUTE.}}$ SINCE $\angle A$ OBTUSE (DOT PRODUCT < 0), BOTH $\angle B$ + $\angle C$ MUST BE ACUTE.

(Total 11 marks)

---

81. Parallelogram ABCD is given by the coordinates: $A(-5,1,2)$, $B(1,3,4)$, and $D(-3,3,-2)$.
Let O represent the origin and P the intersection of the diagonals of ABCD.



(a) Find the coordinates of point C.

$\boxed{C(3,5,0)}$

O(0,0,0)

Let O represent the origin and P the intersection of the diagonals of ABCD.

(b) Find the magnitude of vector OP.

P MIDPT OF $\overline{AC}$ AND $\overline{BD}$ $\left(\frac{-5+3}{2}, \frac{1+5}{2}, \frac{2+0}{2}\right)$ $P(-1,3,1)$

$$\overrightarrow{OP} = \begin{pmatrix}-1\\3\\1\end{pmatrix} \quad |\overrightarrow{OP}| = \sqrt{1+9+1} = \boxed{\sqrt{11}}$$

(c) Find $\overrightarrow{PB}\bullet\overrightarrow{CP}$.

$$\overrightarrow{PB}\cdot\overrightarrow{CP} = \overrightarrow{PB}\cdot -\overrightarrow{PC}$$

$$\overrightarrow{PB} = \begin{pmatrix}2\\0\\3\end{pmatrix} \quad \overrightarrow{PC} = \begin{pmatrix}4\\2\\-1\end{pmatrix}$$

$$\begin{pmatrix}2\\0\\3\end{pmatrix}\cdot\begin{pmatrix}-4\\-2\\+1\end{pmatrix} = -8+0+3 = \boxed{-5}$$

(d) Are the diagonals of ABCD perpendicular to each other? Briefly explain.

$$\overrightarrow{AC} = \begin{pmatrix}8\\4\\-2\end{pmatrix} \quad \overrightarrow{BD} = \begin{pmatrix}-4\\0\\-6\end{pmatrix}$$

$$\begin{pmatrix}8\\4\\-2\end{pmatrix}\cdot\begin{pmatrix}-4\\0\\-6\end{pmatrix} = -32+0+12 = \boxed{-20}$$ NO. DOT PRODUCT $\neq 0$.

(Total 7 marks)

82. Find the angle formed by the lines $\underline{\overset{l_1}{\dfrac{x-3}{2} = 2y = \dfrac{1-z}{7}}}$ and $\underline{\overset{l_2}{x = 2y = z + 5}}$.

$\theta = \cos^{-1}\left(\left|\dfrac{u \cdot v}{|u||v|}\right|\right)$

$x = 3 + 2a$ $\quad$ $x = 0 + B$

$y = 0 + \frac{1}{2}a$ $\quad$ $y = 0 + \frac{1}{2}B$

$z = 1 - 7a$ $\quad$ $z = -5 + B$

$l_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + a\begin{pmatrix} 4 \\ 1 \\ -14 \end{pmatrix}$ $\quad$ $l_2 = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} + B\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

SCALED UP ×2 $\quad$ SCALED UP ×2

$u = \begin{pmatrix} 4 \\ 1 \\ -14 \end{pmatrix}$ $\quad$ $v = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$|u| = \sqrt{16 + 1 + 196} = \sqrt{213}$

$|v| = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

$u \cdot v = 8 + 1 - 28 = -19$

$\theta = \cos^{-1}\left(\left|\dfrac{-19}{\sqrt{213}\sqrt{9}}\right|\right)$ $\quad$ $\boxed{\theta = 64.28^\circ}$

---

83. Given that $\boldsymbol{a} = 2\sin\theta\,\boldsymbol{i} + (1 - \sin\theta)\boldsymbol{j}$, find the value of the acute angle θ, so that $\boldsymbol{a}$ is perpendicular to the line $\underline{x + y = 1}$.

$y = -x + 1$

$a = \begin{pmatrix} 2\sin\theta \\ 1 - \sin\theta \end{pmatrix}$ $\quad$ $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2\sin\theta \\ 1 - \sin\theta \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\sin\theta - (1 - \sin\theta) = 0$

$2\sin\theta - 1 + \sin\theta = 0$

$3\sin\theta = 1$

$\theta = \sin^{-1}\left(\frac{1}{3}\right)$

$\boxed{\theta = 19.47^\circ}$

(Total 5 marks)

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84. Find the angle formed by the line $x = y = z$ and the plane containing the 3 points below:

$A(1, 2, -1)$ $\quad$ $B(2, 4, 5)$ $\quad$ $C(0, -4, 1)$

PLANE

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ $\quad$ $\overrightarrow{AC} = \begin{pmatrix} -1 \\ -6 \\ 2 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 4 - (-36) \\ -6 - 2 \\ -6 - (-2) \end{pmatrix} = \begin{pmatrix} 40 \\ -8 \\ -4 \end{pmatrix}$

$n = \begin{pmatrix} 40 \\ -8 \\ -4 \end{pmatrix}$

$|u| = \sqrt{3}$

$|n| = \sqrt{1600 + 64 + 16} = \sqrt{1680}$

LINE

$l_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$u \cdot n = 40 - 8 - 4 = 28$

$\theta = \sin^{-1}\left(\left|\dfrac{28}{\sqrt{3}\sqrt{1680}}\right|\right)$

$\boxed{\theta = 23.23^\circ}$

85. The position vector of point A is $2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and the position vector of point B is $4\boldsymbol{i} - 5\boldsymbol{j} + 21\boldsymbol{k}$.

(a) Find the unit vector $\boldsymbol{u}$ in the direction of $\overrightarrow{AB}$.

$$\vec{AB} = \begin{pmatrix} 2 \\ -8 \\ 20 \end{pmatrix} \qquad |\vec{AB}| = \sqrt{4 + 64 + 400} = \sqrt{468} = 6\sqrt{13} \qquad u = \frac{1}{6\sqrt{13}}\begin{pmatrix} 2 \\ -8 \\ 20 \end{pmatrix}$$

$$u = \frac{1}{\sqrt{13}}\begin{pmatrix} 1/3 \\ -4/3 \\ 10/3 \end{pmatrix}$$

(b) Show that $\boldsymbol{u}$ is perpendicular to $\overrightarrow{OA}$.

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 1/3 \\ -4/3 \\ 10/3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{13}}\left(\frac{2}{3} - \frac{12}{3} + \frac{10}{3}\right) = \boxed{0} \checkmark \text{ PERPENDICULAR}$$

(Total 4 marks)

---

86. The line L_1 is represented by $r_1 = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and the line L_2 by $r_2 = \begin{pmatrix} 3 \\ -3 \\ 8 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$.

The lines L_1 and L_2 intersect at point T. Find the coordinates of T.

r_1: $x = 2 + 1s$, $y = 5 + 2s$, $z = 3 + 3s$

r_2: $x = 3 - 1t$, $y = -3 + 3t$, $z = 8 - 4t$

$2 + 1s = 3 - 1t$

$s + t = 1$

$s = 1 - t$

$5 + 2s = -3 + 3t$

$2s - 3t = -8$

$2(1-t) - 3t = -8$

$2 - 2t - 3t = -8$

$-5t = -10$

$t = 2$ / $s = -1$

$x = 1$ $y = 3$ $z = 0$

$\boxed{T(1, 3, 0)}$

(Total 6 marks)

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87. A ray of light coming from the point $(-1, 3, 2)$ is travelling in the direction of vector $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and meets the plane $\pi : x + 3y + 2z - 24 = 0$. Find the angle that the ray of light makes with the plane.

RAY

$$l = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + a\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

PLANE

$x + 3y + 2z = 24$

$$n = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$$u = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \qquad n = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$u \cdot n = 4 + 3 - 4 = 3$

$|u| = \sqrt{16 + 1 + 4} = \sqrt{21}$

$|n| = \sqrt{1 + 9 + 4} = \sqrt{14}$

$$\sin\theta = \frac{3}{\sqrt{21} \cdot \sqrt{14}}$$

$\boxed{\theta = 10.08^\circ}$

(Total 6 marks)

88. The vector equation of line l is given as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

Find the Cartesian equation of the plane containing the line l and the point A(4, –2, 5).

$x = 1 - 1\lambda$
$y = 3 + 2\lambda$
$z = 6 - 1\lambda$

$A(1, 3, 6)$
$B(4, -2, 5)$

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$

$l_1 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5-(-2) \\ 1-(-3) \\ 6-5 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}$

$7x + 4y + 1z = d$
$7(4) + 4(-2) + 1(5) = d$
$28 - 8 + 5 = d$
$d = 25$

$\boxed{7x + 4y + 1z = 25}$

(Total 6 marks)

---

89. The points A, B, C have position vectors $\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $3\boldsymbol{i} + \boldsymbol{k}$ respectively and lie in the plane π.

(a) Find

$A\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $B\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $C\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

(i) the area of the triangle ABC;

$A = \frac{1}{2} \left| \overrightarrow{AB} \times \overrightarrow{AC} \right|$

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1-(-1) \\ 2-0 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$

$\left| \overrightarrow{AB} \times \overrightarrow{AC} \right| = \sqrt{0+4+4} = 2\sqrt{2}$

$A = \frac{1}{2} \cdot 2\sqrt{2} = \boxed{\sqrt{2}}$

(ii) the Cartesian equation of the plane π.

$n = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$

$0x + 2y - 2z = d$
$0(1) + 2(1) - 2(2) = d$
$d = -2$

$2y - 2z = -2$

$\boxed{y - z = -1}$

The line L passes through the origin and is normal to the plane π, it intersects π at the point D.

(b) Find

(i) the coordinates of the point D;

$L = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$

$x = 0$
$y = 2\lambda$
$z = -2\lambda$

$2\lambda - (-2\lambda) = -1$
$4\lambda = -1$
$\lambda = -\frac{1}{4}$

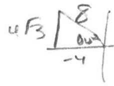
$x = 0$
$y = -\frac{1}{2}$
$z = \frac{1}{2}$

$\boxed{D(0, -\frac{1}{2}, \frac{1}{2})}$

(ii) the distance of π from the origin.

$D = \sqrt{0 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \boxed{\frac{\sqrt{2}}{2}}$

(Total 11 marks)



$z_1 = 8\text{cis}\left(\frac{2\pi}{3}\right)$ $z_1^* = 8\text{cis}\left(\frac{4\pi}{3}\right)$ $-z_2 = -3\text{cis}\left(\frac{\pi}{4}\right)$

90. Given the complex numbers $z_1 = -4 + 4i\sqrt{3}$ and $z_2 = 3\ \text{cis}\left(\frac{\pi}{4}\right)$, find the following. Express all answers in modulus-argument form where $r > 0$ and $0 \le \theta < 2\pi$. All angles must be expressed in radians.

a.) $(z_1)\cdot(z_2)$

$8\text{cis}\left(\frac{2\pi}{3}\right)\cdot 3\text{cis}\left(\frac{\pi}{4}\right)$

$24\text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$

$\boxed{24\text{cis}\left(\frac{11\pi}{12}\right)}$

b.) $(z_1^*)\cdot(-z_2)$

$8\text{cis}\left(\frac{4\pi}{3}\right)\cdot -3\text{cis}\left(\frac{\pi}{4}\right)$

$-24\text{cis}\left(\frac{4\pi}{3} + \frac{\pi}{4}\right)$

$-24\text{cis}\left(\frac{19\pi}{12}\right)$

$\boxed{24\text{cis}\left(\frac{7\pi}{12}\right)}$

c.) $\left(\frac{z_1}{z_2}\right)^*$

$\dfrac{8\text{cis}\left(\frac{2\pi}{3}\right)}{3\text{cis}\left(\frac{\pi}{4}\right)}$

$\left(\frac{8}{3}\text{cis}\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)\right)^*$

$\left(\frac{8}{3}\text{cis}\left(\frac{5\pi}{12}\right)\right)^*$

$\frac{8}{3}\text{cis}\left(-\frac{5\pi}{12}\right) = \boxed{\frac{8}{3}\text{cis}\left(\frac{19\pi}{12}\right)}$

---

91. Given that $\frac{z}{z+2} = 2 - \text{i}$, $z \in \mathbb{C}$, find z in the form $a + \text{i}b$.

SKIP.... REPEAT OF #70

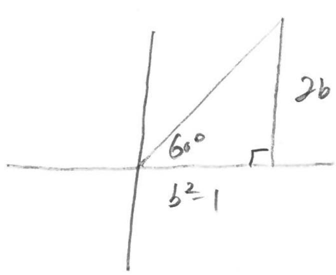
(Total 4 marks)

---

92. Given that $z = (b + \text{i})^2$, where b is real and positive, find the value of b when $\arg z = 60°$.

$z = b^2 + i^2 + 2bi$

$z = (b^2 - 1) + 2bi$



$\tan 60° = \frac{2b}{b^2-1}$

$\sqrt{3} = \frac{2b}{b^2-1}$

$(b^2-1)\sqrt{3} = 2b$

$\sqrt{3}\cdot b^2 - 2b - \sqrt{3} = 0$

$(\sqrt{3}b + 1)(b - \sqrt{3}) = 0$

$b - 3b$

$b = -\frac{1}{\sqrt{3}}, \sqrt{3}$

$\boxed{b = \sqrt{3}}$

(Total 6 marks)

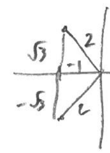
93. The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β?

(a) Find α and β in the form $re^{i\theta}$. **(6)**

$$z = \frac{-2 \pm \sqrt{4-4(1)(4)}}{2}$$

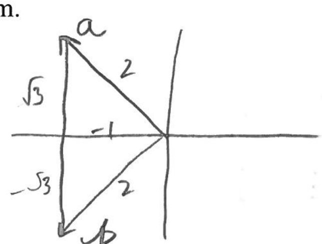
$$z = \frac{-2 \pm 2\sqrt{3}}{2}$$

$$z = -1 \pm \sqrt{3}$$



$$\boxed{\begin{aligned} a &= 2\,\mathrm{cis}\left(\tfrac{2\pi}{3}\right) = 2e^{i\cdot\frac{2\pi}{3}} \\ b &= 2\,\mathrm{cis}\left(\tfrac{4\pi}{3}\right) = 2e^{i\cdot\frac{4\pi}{3}} \end{aligned}}$$

(b) Given that α lies in the second quadrant of the Argand diagram, mark α and β on an Argand diagram. **(2)**



(c) Using De Moivre's theorem find $\dfrac{\alpha^3}{\beta^2}$ in the form $a + ib$. **(4)**

$$\frac{\left[2\,\mathrm{cis}\left(\frac{2\pi}{3}\right)\right]^3}{\left[2\,\mathrm{cis}\left(\frac{4\pi}{3}\right)\right]^2} = \frac{8\,\mathrm{cis}\left(3\cdot\frac{2\pi}{3}\right)}{4\,\mathrm{cis}\left(2\cdot\frac{4\pi}{3}\right)} = 2\,\mathrm{cis}\left(2\pi - \frac{8\pi}{3}\right) = 2\,\mathrm{cis}\left(-\frac{2\pi}{3}\right) = 2\,\mathrm{cis}\left(\frac{4\pi}{3}\right) = \boxed{B}$$

(d) Using De Moivre's theorem or otherwise, show that $\alpha^3 = \beta^3$. **(3)**

$a^3 = 8\,\mathrm{cis}\,(2\pi) = 8\,\mathrm{cis}\,0$ (SEE ABOVE)

$b^3 = \left[2\,\mathrm{cis}\left(\frac{4\pi}{3}\right)\right]^3 = 8\,\mathrm{cis}\left(3\cdot\frac{4\pi}{3}\right) = 8\,\mathrm{cis}\,0$ ✓

(Total 15 marks)

---

94. Given that $|z| = \sqrt{10}$, solve the equation $5z + \dfrac{10}{z^*} = 6 - 18i$, where z^* is the conjugate of z.

$\times\ z^*$

$z = a + bi$

$z^* = a - bi$

$|z| = \sqrt{a^2 + b^2} = \sqrt{10}$

$a^2 + b^2 = 10$

$5(a+bi)(a-bi) + 10 = (6-18i)(a-bi)$

$5(a^2+b^2) + 10 = (6-18i)(a-bi)$

$60 = (6-18i)(a-bi)$

$a - bi = \dfrac{60}{6-18i}$

$a - bi = \dfrac{10}{1-3i}\dfrac{(1+3i)}{(1+3i)}$

$= \dfrac{10+30i}{1+9}$

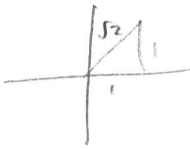
$a - bi = 1 + 3i$

$a + bi = 1 - 3i$

$\boxed{z = 1 - 3i}$

(Total 7 marks)

95. (a) Express the complex number $1 + \mathrm{i}$ in the form $\sqrt{a}e^{i\frac{\pi}{b}}$, where $a, b \in \mathbb{Z}^+$. **(2)**



$= \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right) = \boxed{\sqrt{2}e^{i\cdot\frac{\pi}{4}}}$

(b) Using the result from (a), show that $\left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^n$, where $n \in \mathbb{Z}$, has only eight distinct values. **(5)**

$\left(\frac{1+i}{\sqrt{2}}\right)^n = \left(\frac{\sqrt{2}e^{i\cdot\frac{\pi}{4}}}{\sqrt{2}}\right)^n = \left(e^{i\cdot\frac{\pi}{4}}\right)^n = 1^n e^{i\cdot\frac{\pi}{4}\cdot n} = e^{in\frac{\pi}{4}}$

$\boxed{n\cdot\frac{\pi}{4} = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}}$ $2\pi \ldots$ REPEATS

$\downarrow$
$= 0$

(c) **Hence** solve the equation $z^8 - 1 = 0$. **(2)**

$z^8 = 1$

$z^8 = 1 + 0i$

$z^8 = 1\,\mathrm{cis}(0)$

$\sqrt[8]{1}\,\mathrm{cis}\left(\frac{0 + 2\pi k}{8}\right), \quad k = 0, \ldots, 7$

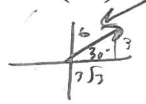
$z = 1\,\mathrm{cis}\left(0 + \frac{\pi}{4}\cdot k\right)$

$z = 1\,\mathrm{cis}(n), \quad n = 0, \frac{\pi}{4}, \ldots, \frac{7\pi}{4}$

(Total 9 marks)

96. Find the following sum, expressing your answer in modulus-argument form.

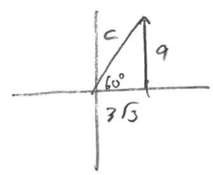
$$6\mathrm{cis}\left(\frac{\pi}{2}\right) + 6\mathrm{cis}\left(\frac{\pi}{6}\right)$$



$z_1 = 0 + 6i$

$z_2 = 3\sqrt{3} + 3i$

$z_1 + z_2 = 3\sqrt{3} + 9i$



$3\sqrt{3}\cdot\sqrt{3} = 9$

30 – 60 – 90

$x \quad x\sqrt{3} \quad 2x$

$(3\sqrt{3})^2 + 9^2 = c^2$

$27 + 81 = c^2$

$c^2 = 108$

$c = 6\sqrt{3}$

$\boxed{6\sqrt{3}\,\mathrm{cis}\left(\frac{\pi}{3}\right)}$

(Total 3 marks)

97. Consider the complex number $\omega = \frac{z + \mathrm{i}}{z + 2}$, where $z = x + \mathrm{i}y$. If $\omega = \mathrm{i}$, determine z in the form $z = r\,\mathrm{cis}\,\theta$.

$i = \frac{z+i}{z+2}$

$i(z+2) = z + i$

$iz + 2i = z + i$

$iz - z = -i$

$z(-1+i) = -i$

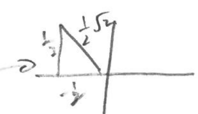
$z = \frac{-i}{-1+i}$

$z = \frac{-i(-1-i)}{(-1+i)(-1-i)}$

$z = \frac{-1+i}{1-i^2}$

$z = \frac{-1+i}{2}$

$z = -\frac{1}{2} + \frac{1}{2}i$



$\boxed{z = \frac{\sqrt{2}}{2}\mathrm{cis}\left(\frac{3\pi}{4}\right)}$

(Total 6 marks)

98. Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$

(a) Find an expression for z, the common ratio of this series. **(2)**

$\frac{\frac{1}{2}e^{2i\theta}}{e^{i\theta}} = \boxed{\frac{1}{2}e^{i\theta}}$ $\quad \frac{1}{2}\text{cis}\,\theta$

(b) Show that $|z| < 1$. **(2)**

$\left|\frac{1}{2}\text{cis}\,\theta\right| = \boxed{\frac{1}{2} < 1}$

(c) Write down an expression for the sum to infinity of this series. **(2)**

$a_1 = e^{i\theta}$

$r = \frac{1}{2}\text{cis}\,\theta$ $\longrightarrow \boxed{S = \frac{e^{i\theta}}{1 - \frac{1}{2}e^{i\theta}}}$

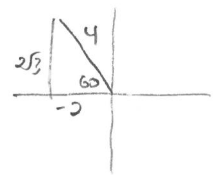
(Total 6 marks)

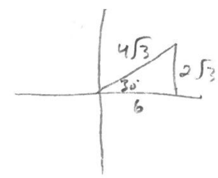
---

99. The complex number z is defined by

$4\text{cis}\,\frac{2\pi}{3} + 4\sqrt{3}\,\text{cis}\,\frac{\pi}{6}$

$$z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + 4\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$



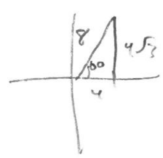


$-C^{+}_{S}$

(a) Express z in the form $re^{i\theta}$, where r and θ have exact values.

$z = -2 + 2\sqrt{3}i + 6 + 2\sqrt{3}i$

$z = 4 + 4\sqrt{3}i$



$z = 8\,\text{cis}\,\frac{\pi}{3}$

$\boxed{z = 8e^{i\cdot\frac{\pi}{3}}}$ $\quad r = 8$, $\theta = \frac{\pi}{3}$

(b) Find the cube roots of z, expressing in the form $re^{i\theta}$, where r and θ have exact values.

$\sqrt[3]{z} = \sqrt[3]{8\,\text{cis}\,\frac{\pi}{3}} = \sqrt[3]{8}\,\text{cis}\left(\frac{\frac{\pi}{3} + 2\pi n}{3}\right) \quad n = 0, 1, 2$

$= 2\,\text{cis}\left(\frac{\pi}{9} + \frac{2\pi}{3}\cdot n\right)$ $\quad \frac{6\pi}{9}$

$\boxed{\sqrt[3]{z} = 2\,\text{cis}\,\frac{\pi}{9},\ 2\,\text{cis}\left(\frac{7\pi}{9}\right),\ 2\,\text{cis}\left(\frac{13\pi}{9}\right)}$

(Total 6 marks)

---

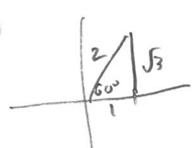
100. Let $z_1 = r\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and $z_2 = 1 + \sqrt{3}\,i$.

$z_1 = r\,\text{cis}\,\frac{\pi}{4}$

(a) Write z_2 in modulus-argument form.

(b) Find the value of r if $|z_1 z_2^3| = 2$.

(a)



$\boxed{z_2 = 2\,\text{cis}\,\frac{\pi}{3}}$

(b) $(z_2)^3 = 2^3\,\text{cis}\left(3\cdot\frac{\pi}{3}\right) = 8\,\text{cis}(\pi)$

$z_1(z_2)^3 = r\,\text{cis}\left(\frac{\pi}{4}\right)\cdot 8\,\text{cis}(\pi) = 8r\,\text{cis}\left(\frac{5\pi}{4}\right)$

$8r = 2$

$\boxed{r = \frac{1}{4}}$

(Total 6 marks)

101. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \text{i}$. **(6)**

$z^4 = \sqrt{2}\,\text{cis}\left(\frac{7\pi}{4}\right)$

$z = \sqrt[4]{\sqrt{2}\,\text{cis}\left(\frac{7\pi}{4}\right)}$

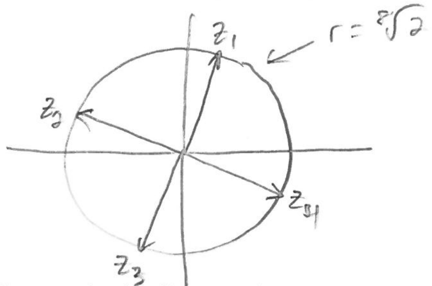
$z = \sqrt[8]{2} \cdot \text{cis}\left(\frac{\frac{7\pi}{4} + 2\pi \cdot n}{4}\right)$ $\quad n = 0, 1, 2, 3$

$z = \sqrt[8]{2}\,\text{cis}\left(\frac{7\pi}{16} + \underbrace{\frac{\pi}{2}\cdot n}_{\frac{8\pi}{16}}\right)$

$$\boxed{\begin{aligned} z_1 &= \sqrt[8]{2}\,\text{cis}\left(\tfrac{7\pi}{16}\right) \\ z_2 &= \sqrt[8]{2}\,\text{cis}\left(\tfrac{15\pi}{16}\right) \\ z_3 &= \sqrt[8]{2}\,\text{cis}\left(\tfrac{23\pi}{16}\right) \\ z_4 &= \sqrt[8]{2}\,\text{cis}\left(\tfrac{31\pi}{16}\right) \end{aligned}}$$

(78.75)
(168.75)
(258.75)
(348.75)

(b) Draw these roots on an Argand diagram. **(2)**



(b) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\frac{z_2}{z_1}$ in the form $a + \text{i}b$. **(4)**

$\frac{z_2}{z_1} = \frac{\sqrt[8]{2}\,\text{cis}\left(\frac{15\pi}{16}\right)}{\sqrt[8]{2}\,\text{cis}\left(\frac{7\pi}{16}\right)} = 1\,\text{cis}\left(\frac{15\pi}{16} - \frac{7\pi}{16}\right) = \boxed{1\,\text{cis}\left(\frac{\pi}{2}\right)}$

(Total 12 marks)

---

102. Let $f(x) = 6\sqrt[3]{x^2}$. Find $f'(x)$.

$f(x) = 6x^{2/3}$

$\boxed{f'(x) = 4x^{-1/3}}$

(Total 6 marks)

---

103. Let $f(x) = x^3 - 2x^2 - 1$.

(a) Find $f'(x)$.

(b) Find the gradient of the curve of $f(x)$ at the point $(2, -1)$.

(a) $f'(x) = 3x^2 - 4x$

(b) $f'(2) = 3(2)^2 - 4(2)$ $\quad \boxed{f'(2) = 4}$

(Total 6 marks)

---

104. A gradient function is given by $\frac{\text{d}y}{\text{d}x} = 10\text{e}^{2x} - 5$. When $x = 0$, $y = 8$. Find the value of y when $x = 1$.

$y = 5e^{2x} - 5x + c$

$8 = 5e^0 - 5(0) + c$

$c = 3$

$y = 5e^{2x} - 5x + 3$

$y = 5e^2 - 5(1) + 3$

$\boxed{y = 5e^2 - 2}$

(Total 8 marks)

105. Let $f(x) = kx^4$. The point P(1, k) lies on the curve of f. At P, the normal to the curve is parallel to $y = -\frac{1}{8}x$

Find the value of k.

$f'(x) = 4kx^3$

$f'(1) = 4k$ TANGENT

NORMAL $= -\frac{1}{8}$

TANG $= 8$

$4k = 8$ $\boxed{k = 2}$

(Total 6 marks)

---

106. The curve C has equation $y = \frac{1}{8}(9 + 8x^2 - x^4)$.

(a) Find the coordinates of the points on C at which $\frac{dy}{dx} = 0$. **(4)**

$\frac{dy}{dx} = \frac{1}{8}(16x - 4x^3) = 2x - \frac{1}{2}x^3$

$2x - \frac{1}{2}x^3 = 0$

$4x - x^3 = 0$

$x(4 - x^2) = 0$

$x = 0, \pm 2$

$(0, \frac{9}{8})$

$(2, \frac{25}{8})$

$(-2, \frac{25}{8})$

(b) The tangent to C at the point P(1, 2) cuts the x-axis at the point T. Determine the coordinates of T. **(4)**

$\frac{dy}{dx} = \frac{1}{8}(16 - 4) = \frac{3}{2}$

$2 = \frac{3}{2}(1) + b$

$b = \frac{1}{2}$

$y = \frac{3}{2}x + \frac{1}{2}$

X-INT

$y = 0$

$\frac{3}{2}x + \frac{1}{2} = 0$

$x = -1/3$

$T(-\frac{1}{3}, 0)$

(c) The normal to C at the point P cuts the y-axis at the point N. Find the area of triangle PTN. **(7)**

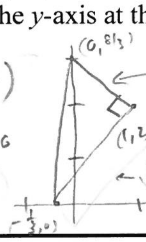
NORMAL $= -\frac{2}{3}$

$2 = -\frac{2}{3}(1) + b$

$b = \frac{8}{3}$

$N(0, \frac{8}{3})$

NORMAL $\perp$ TANG



$\sqrt{1 + \frac{4}{9}}$

$\frac{\sqrt{13}}{3}$

$\sqrt{4 + \frac{13}{9}}$

$\frac{\sqrt{52}}{3} = \frac{2\sqrt{13}}{3}$

$A = \frac{1}{2} \cdot \frac{\sqrt{13}}{3} \cdot \frac{2\sqrt{13}}{3}$

$\boxed{A = \frac{13}{9}}$

(Total 15 marks)

---

107. Consider the function $f(x) = 3x^2 - 5x + k$.

(a) Write down $f'(x)$. $f'(x) = 6x - 5$

The equation of the tangent to the graph of f at $x = p$ is $y = 7x - 9$.

SLOPE

(b) Find the values of p and k.

$6x - 5 = 7$

$6x = 12$

$x = 2$

$y = 7(2) - 9$

$y = 5$

$(2, 5)$

$5 = 3(2)^2 - 5(2) + k$

$5 = 12 - 10 + k$

$k = 3$

$\boxed{p = 2, k = 3}$

(Total 6 marks)

---

108. Consider the curve with equation $f(x) = px^2 + qx$, where p and q are constants. The point A(1, 3) lies on the curve. The tangent to the curve at A has gradient 8. Find the values of p and q.

$3 = p(1)^2 + q(1)$

$3 = p + q$

$f'(x) = 2px + q$

$8 = 2p(1) + q$

$8 = 2p + q$

$-3 = -p - q$

$8 = 2p + q$

$5 = p$

$3 = 5 + q$

$q = -2$

$\boxed{p = 5, \; q = -2}$

(Total 7 marks)

109. Let $f(x) = \dfrac{3x^2}{5x-1}$. QUOTIENT RULE $\dfrac{\text{lodhi} - \text{hidlo}}{\text{lo}^2}$

(a) Write down the **equation** of the vertical asymptote of $y = f(x)$.

$\boxed{x = \frac{1}{5}}$

(1)

(b) Find $f'(x)$. Give your answer in the form $\dfrac{ax^2 + bx}{(5x-1)^2}$ where a and $b \in \mathbb{Z}$.

$f'(x) = \dfrac{(5x-1)(6x) - (3x^2)(5)}{(5x-1)^2} = \dfrac{30x^2 - 6x - 15x^2}{(5x-1)^2} = \boxed{\dfrac{15x^2 - 6x}{(5x-1)^2}}$

(4)

(Total 5 marks)

110. For what values of m is the line $y = mx + 5$ a tangent to the parabola $y = 4 - x^2$?

$y = 4 - x^2$

$\frac{dy}{dx} = -2x$

$m = -2x$

$y = (-2x)x + 5$

$y = -2x^2 + 5$

$-2x^2 + 5 = 4 - x^2$

$x^2 = 1$

$x = \pm 1$

$\boxed{m = \pm 2}$

(Total 3 marks)

111. The line $y = 16x - 9$ is a tangent to the curve $y = 2x^3 + ax^2 + bx - 9$ at the point (1,7). Find the values of a and b. SLOPE

$7 = 2(1)^3 + a(1)^2 + b(1) - 9$

$7 = 2 + a + b - 9$

$a + b = 14$

$\frac{dy}{dx} = 6x^2 + 2ax + b$

$16 = 6(1)^2 + 2a(1) + b$

$2a + b = 10$

$2a + \cancel{b} = 10$

$-a - \cancel{b} = -14$

$\boxed{\begin{aligned} a &= -4 \\ b &= 18 \end{aligned}}$

(Total 3 marks)

112. Consider the function $f(x) = x^3 - 3x^2 - 9x + 10$, $x \in \mathbb{R}$.

(a) Find the equation of the straight line passing through the maximum and minimum points of the graph $y = f(x)$.

(4)

$f'(x) = 3x^2 - 6x - 9$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$x = 3, -1$

$f(3) = -17$

$f(-1) = 15$

$m = \dfrac{15 - (-17)}{-1 - 3} = \dfrac{32}{-4} = -8$

$-17 = -8(3) + b$

$b = 7$

$\boxed{y = -8x + 7}$

(b) Show that the point of inflexion of the graph $y = f(x)$ lies on this straight line.

(2)

$f''(x) = 6x - 6$

$6x - 6 = 0$

$x = 1$

$f(1) = (1)^3 - 3(1)^2 - 9(1) + 10$

$f(1) = -1$

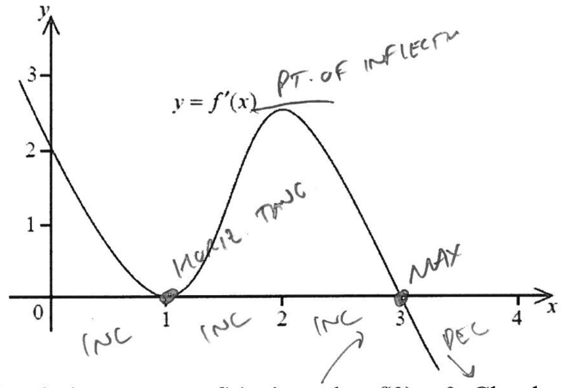
$y = -8(1) + 7$

$y = -1$

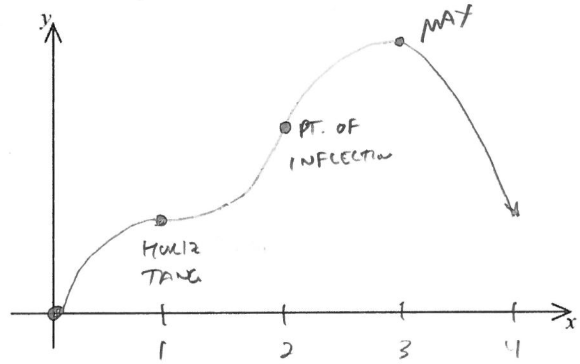
✓

(Total 3 marks)

113. The diagram below shows a sketch of the ***gradient function f′(x)*** of the curve $f(x)$.



On the graph below, sketch the curve $y = f(x)$ given that $f(0) = 0$. Clearly indicate on the graph any maximum, minimum or inflexion points.



$P\left(-3^{-1/5}, -5\cdot 3^{-4/5}\right)$

(Total 5 marks)

114. The function f is given by $f(x) = \dfrac{x^5+2}{x}$, $x \neq 0$. There is a point of inflexion on the graph of f at the point P. Find the coordinates of P.

MAY USE QUOTIENT RULE

$f(x) = x^4 + 2x^{-1}$

$f'(x) = 4x^3 - 2x^{-2}$

$f''(x) = 12x^2 + 4x^{-3}$

$\to 12x^2 + 4x^{-3} = 0$

$\cancel{4x^{-3}}(3x^5+1) = 0$

$x^5 = -\frac{1}{3}$

$x = \frac{-1}{\sqrt[5]{3}}$ or $-3^{-1/5}$

$f(-3^{-1/5}) = (-3^{-1/5})^4 + 2(-3^{-1/5})^{-1}$

$= 3^{-4/5} - 2\cdot 3^{1/5}$

$= 3^{-4/5}(1 - 2\cdot 3)$

$= -5\cdot 3^{-4/5}$

(Total 6 marks)

115. The displacement s meters of a moving body B from a fixed point O at time t seconds is given by

$$s = 50t - 10t^2 + 1000.$$

$d \to v \to a$

(a) Find the velocity of B in m s^{-1}.

(b) Find its maximum displacement from O.

(a) $\frac{ds}{dt} = \boxed{v = 50 - 20t}$

$S(2.5) = 50(2.5) - 10(2.5)^2 + 1000 = 1062.5$

$S(0) = 1000$

(b) $50 - 20t = 0$

$t = 2.5$

$\boxed{62.5\text{ m}}$

(AT 1000 $t=0$, AT 1062.5 $t=2.5$)

(Total 6 marks)

116. The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$. $(3, 5)$

(a) Find the value of p and the value of q. **(4)**

$5 = p + q(3) - (3)^2$

$p + 3q = 14$

$f'(x) = q - 2x$

$0 = q - 2(3)$

$\boxed{q = 6}$

$p + 18 = 14$

$\boxed{p = -4}$

$f(x) = -4 + 6x - x^2$

$\boxed{f(x) = -x^2 + 6x - 4}$

(b) The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph. $g(x)$ $\quad 3 \rightarrow$ **(2)**

$f(x) = -x^2 + 6x - 4$

$f(x) = -(x^2 - 6x + 4)$

$f(x) = -(x^2 - 6x + 9) + 5$

$f(x) = -(x-3)^2 + 5 \quad V(3,5)$

$V(6,5)$

$g(x) = -(x-6)^2 + 5$

$= -(x^2 - 12x + 36) + 5$

$= \boxed{-x^2 + 12x - 31}$

(Total 6 marks)

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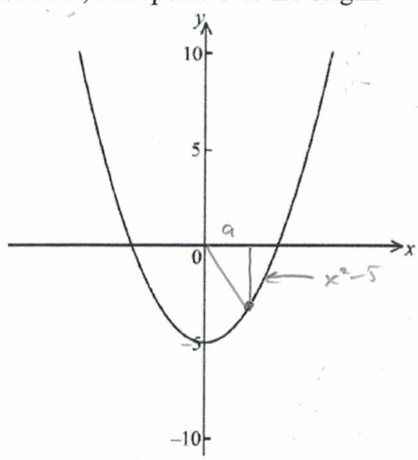
117. The curve $y = x^2 - 5$ is shown below, with point O at the origin.



A point P on the curve has x-coordinate equal to a.

(a) Show that the distance OP is $\sqrt{a^4 - 9a^2 + 25}$. **(2)**

$D = \sqrt{a^2 + (a^2-5)^2}$

$= \sqrt{a^2 + a^4 - 10a^2 + 25}$

$\boxed{D = \sqrt{a^4 - 9a^2 + 25}}$

(b) Find the values of a for which the curve is closest to the origin. **(5)**

$D' = \frac{1}{2}(a^4 - 9a^2 + 25)^{-1/2} \cdot (4a^3 - 18a)$

$D' = \dfrac{4a^3 - 18a}{2\sqrt{a^4 - 9a^2 + 25}}$

$4a^3 - 18a = 0$

$2a(2a^2 - 9) = 0$

$a = 0, \pm \frac{3}{\sqrt{2}}$

$-4 \quad -\frac{3}{\sqrt{2}} \quad -1 \quad 0 \quad 1 \quad \frac{3}{\sqrt{2}} \quad 4$

$- \quad + \quad - \quad +$

MIN MAX MIN

$\boxed{a = \pm \frac{3}{\sqrt{2}}}$

(Total 7 marks)

---

118. Find the equation of the normal to the curve $5xy^2 - 2x^2 = 18$ at the point $(1, 2)$.

$5y^2 + 5x \cdot 2y \cdot \frac{dy}{dx} - 4x = 0$

$5(2)^2 + 10(1)(2) \cdot \frac{dy}{dx} - 4(1) = 0$

$20 + 20 \cdot \frac{dy}{dx} - 4 = 0$

$\frac{dy}{dx} = \frac{4-20}{20}$

$\boxed{\frac{dy}{dx} = -\frac{4}{5}}$

NORMAL $= \frac{5}{4}$

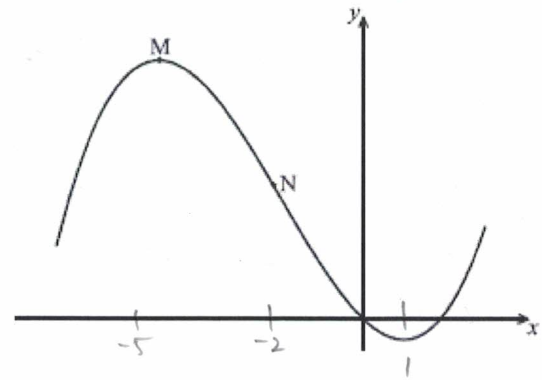
$2 = \frac{5}{4}(1) + b$

$b = \frac{3}{4}$

$\boxed{y = \frac{5}{4}x + \frac{3}{4}}$

(Total 7 marks)

119. Consider $f(x) = \frac{1}{3}x^3 + 2x^2 - 5x$. Part of the graph of f is shown below. There is a maximum point at M, and a point of inflection at N.



(a) Find $f'(x)$. **(2)**

$f'(x) = x^2 + 4x - 5 = (x+5)(x-1)$

(b) Find the coordinate of M. **(3)**

$(x+5)(x-1) = 0$

$x = -5, 1$ $\quad f(-5) = \frac{100}{3}$ $\quad \boxed{M\left(-5, \frac{100}{3}\right)}$

(c) Find the coordinate of N. **(2)**

$f''(x) = 2x + 4$

$2x + 4 = 0$

$x = -2$ $\quad f(-2) = \frac{46}{3}$ $\quad \boxed{N\left(-2, \frac{46}{3}\right)}$

(d) The line L is the tangent to the curve of f at (3, 12). Find the equation of L in the form $y = ax + b$. **(3)**

$f'(3) = 3^2 + 4(3) - 5 = 16$

$m = 16$

$12 = 16(3) + b$

$b = -36$

$\boxed{y = 16x - 36}$

(Total 10 marks)

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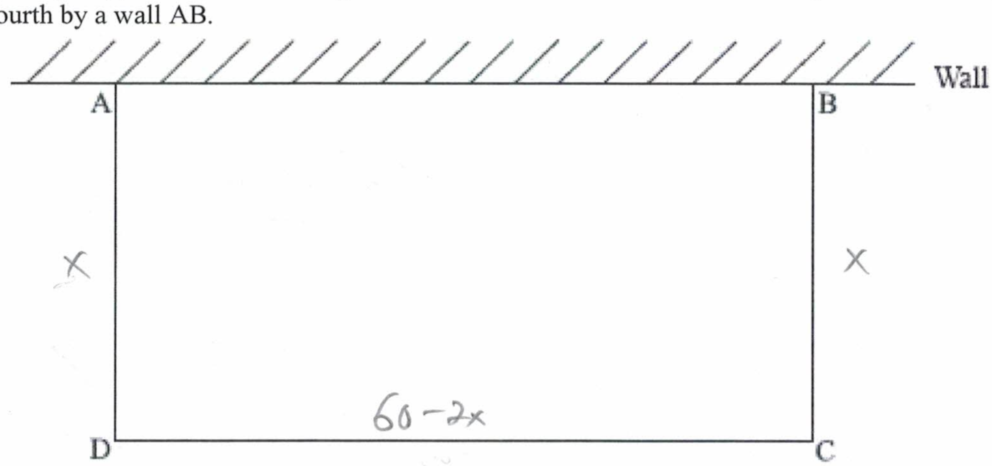
120. The following diagram shows a rectangular area ABCD enclosed on three sides by 60 m of fencing, and on the fourth by a wall AB.



Find the dimensions of the rectangle that gives its maximum area.

$A(x) = x(60 - 2x)$

$A(x) = 60x - 2x^2$

$A'(x) = 60 - 4x = 0$

$x = 15$

$60 - 2(15) = 30$

+20 $\quad$ -20

10 $\quad$ 15 $\quad$ 20

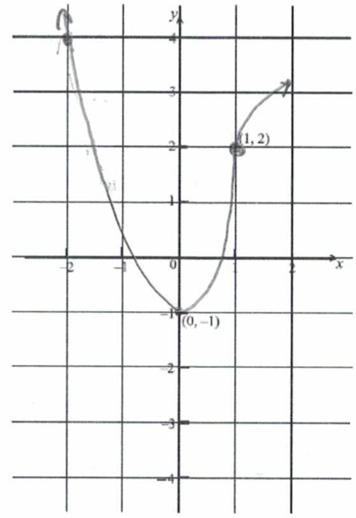
MAX

$\boxed{15\text{m} \times 30\text{m}}$

(Total 6 marks)

121. On the axes below, sketch a curve $y = f(x)$ which satisfies the following conditions.

x	$f(x)$	$f'(x)$	$f''(x)$
$-2 \le x < 0$		negative	positive
0	-1	0	positive
$0 < x < 1$		positive	positive
1	2	positive	0
$1 < x \le 2$		positive	negative



(Total 6 marks)

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122. Find the equation of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point $(1, -2)$.

$6xy + 3x^2 \cdot \frac{dy}{dx} + 2y^2 + 2x \cdot 2y \cdot \frac{dy}{dx} = 0$

$6xy + 3x^2\left(\frac{dy}{dx}\right) + 2y^2 + 4xy\left(\frac{dy}{dx}\right) = 0$

$6(1)(-2) + 3(1)^2\left(\frac{dy}{dx}\right) + 2(-2)^2 + 4(1)(-2)\left(\frac{dy}{dx}\right) = 0$

$-12 + 3\left(\frac{dy}{dx}\right) + 8 - 8\left(\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx} = \frac{4}{-5}$

Normal $= \frac{5}{4}$

$-2 = \frac{5}{4}(1) + b$

$b = -\frac{13}{4}$

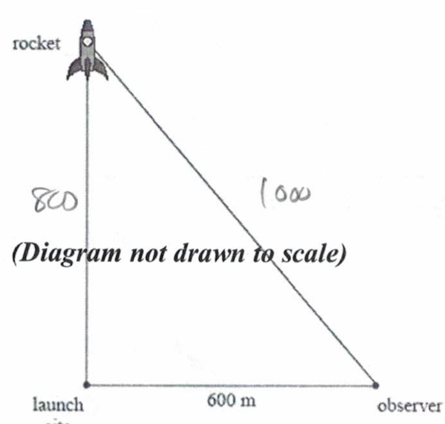
$\boxed{y = \frac{5}{4}x - \frac{13}{4}}$

(Total 7 marks)

---

123. A rocket is rising vertically at a speed of 300 m s^{-1} when it is 800 m directly above the launch site. Calculate the rate of change of the distance between the rocket and an observer, who is 600 m from the launch site and on the same horizontal level as the launch site.

$\frac{dy}{dx} = 300$



(Diagram not drawn to scale)

$600^2 + y^2 = d^2$

$2y \cdot \frac{dy}{dt} = 2d\frac{dd}{dt}$

$\cancel{2}(800)(300) = \cancel{2}(1000) \cdot \frac{dd}{dt}$

$\boxed{\frac{dd}{dt} = 240 \text{ m/s}}$

(Total 6 marks)

---

124. If $f(x) = x - 3x^{\frac{2}{3}}$, $x > 0$,

(a) find the x-coordinate of the point P where $f'(x) = 0$; **(2)**

$f'(x) = 1 - 2x^{-1/3}$ $\quad 1 - 2x^{-1/3} = 0 \quad 2x^{-1/3} = 1 \quad x^{-1/3} = \frac{1}{2} \quad (x^{-1/3})^3 = (2)^3 \quad \underline{x = 8}$

(b) determine and show whether P is a maximum or minimum point. **(3)**

$-\ \searrow \quad \nearrow\ +$

1 $\quad$ 8 $\quad$ 10

(−1) $\quad$ (0.07)

$\boxed{\text{MINIMUM}}$

(Total 5 marks)

---

125. The curve $y = \frac{x^3}{3} - x^2 - 3x + 4$ has a local maximum point at P and a local minimum point at Q. Determine the equation of the straight line passing through P and Q, in the form $ax + by + c = 0$, where a, b, $c \in \mathbb{R}$.

$\frac{dy}{dx} = x^2 - 2x - 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\underline{x = 3, -1}$

$f(3) = -5$

$f(-1) = 17/3$

NO NEED TO DISTINGUISH BETWEEN WHICH IS MAX/MIN

$m = \frac{17/3 - (-5)}{-1-3} = \frac{17/3 + 5}{-4} \cdot \frac{3}{3} = \frac{17+15}{-12} = -8/3$

$-5 = -8/3(3) + b$

$-5 = -8 + b$

$b = 3$

$(y = -8/3x + 3) \cdot 3$

$3y = -8x + 9$

$\boxed{8x + 3y - 9 = 0}$

(Total 6 marks)

---

126. Find the gradient of the curve $e^{xy} + \ln(y^2) + e^y = 1 + e$ at the point (0, 1). IMPLICIT DIFF.

$e^{xy} \cdot \left(y + x \cdot \frac{dy}{dx}\right) + \frac{1}{y^2} \cdot 2y \cdot \frac{dy}{dx} + e^y \cdot \frac{dy}{dx} = 0$ (CONSTANT)

PLUG IN (0, 1) IMMEDIATELY!!

$e^0(1 + 0) + \left(1 \cdot 2 \cdot \frac{dy}{dx}\right) + \left(e^1 \cdot \frac{dy}{dx}\right) = 0$

$1 + 2 \cdot \frac{dy}{dx} + e \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx}(2 + e) = -1$

$\boxed{\frac{dy}{dx} = \frac{-1}{2+e}}$

(Total 7 marks)

---

127. The function f is defined by $f(x) = (\ln(x-2))^2$. Find the coordinates of the point of inflexion of f.

$f'(x) = 2(\ln(x-2)) \cdot \frac{1}{x-2} = \frac{2\ln(x-2)}{x-2}$

$f''(x) = \frac{(x-2) \cdot \frac{2}{x-2} - 2\ln(x-2)(1)}{(x-2)^2} = \frac{2 - 2\ln(x-2)}{(x-2)^2}$

$2 - 2\ln(x-2) = 0$

$2\ln(x-2) = 2$

$\ln(x-2) = 1$

$x - 2 = e$

$\underline{x = e + 2}$

$f(e+2) = 1$

$\boxed{(e+2, 1)}$

(Total 9 marks)