Math HL – Year 1 Name: KEY

IB Math HL Practice Problems Period: _____

**1.** The first three terms of an arithmetic sequence are 7, 9.5, 12.

(a) What is the 41st term of the sequence?

(b) What is the sum of the first 101 terms of the sequence?

(a) $d = 2.5$

$a_n = 7 + 2.5(n-1)$

$a_n = 2.5n + 4.5$

$a_{41} = 2.5(41) + 4.5$

$\boxed{a_{41} = 107}$

(b) $a_1 = 7$ $a_{101} = 257$

$S_{101} = \frac{101}{2}(7 + 257)$

$\boxed{S_{101} = 13{,}332}$

**(Total 4 marks)**

**2.** In the arithmetic series with $n^{th}$ term $u_n$, it is given that $u_4 = 7$ and $u_9 = 22$.
Find the minimum value of $n$ so that $u_1 + u_2 + u_3 + ... + u_n > 10\ 000$.

$d = \frac{22-7}{9-4}$

$d = \frac{15}{5}$

$d = 3$

$S_n = \frac{n}{2}(a_1 + a_n)$

$a_n = a_1 + d(n-1)$

$a_1 + 3d = a_4$

$a_1 = 7 - 9$

$a_1 = -2$

$a_n = -2 + 3(n-1)$

$a_n = -2 + 3n - 3$

$a_n = 3n - 5$

$S_n = \frac{n}{2}(-2 + 3n - 5)$

$10{,}000 = \frac{n}{2}(3n - 7)$

$3n^2 - 7n - 20{,}000 = 0$

$n = \frac{7 \pm \sqrt{49 - 4(3)(-20{,}000)}}{6}$

$n = 82.8$

$\boxed{n = 83}$

**(Total 5 marks)**

**3.** Consider the infinite geometric sequence 3000, – 1800, 1080, – 648, … .

(a) Find the common ratio. **(2)**

(b) Find the 10th term. **(2)**

(c) Find the **exact** sum of the infinite sequence. **(2)**

(a) $r = \frac{-1800}{3000}$

$\boxed{r = -\frac{3}{5}}$

(b) $a_n = 3000 \cdot \left(-\frac{3}{5}\right)^{n-1}$

$a_{10} = 3000 \cdot \left(-\frac{3}{5}\right)^9$

$\boxed{a_{10} = -30.233}$

(c) $S = \frac{a_1}{1-r}$

$S = \frac{3000}{1-\left(-\frac{3}{5}\right)}$

$S = \frac{3000}{\frac{8}{5}}$

$\boxed{S = 1875}$

**(Total 6 marks)**

**4.** The sum of an infinite geometric sequence is $13\frac{1}{2}$, and the sum of the first three terms is 13. Find the first term.

$13\frac{1}{2} = \frac{a_1}{1-r}$ $\quad 13 = \frac{a_1(1-r^3)}{1-r}$

Solve for $\frac{a_1}{1-r}$

$\frac{a_1}{1-r} = 13\frac{1}{2}$ $\quad \frac{a_1}{1-r} = \frac{13}{1-r^3}$

$13\frac{1}{2} = \frac{13}{1-r^3}$

$1-r^3 = \frac{13}{13\frac{1}{2}}$

$1-r^3 = \frac{26}{27}$

$r^3 = \frac{1}{27}$

$r = \frac{1}{3}$

$\frac{a_1}{1-\frac{1}{3}} = 13\frac{1}{2}$

$a_1 = 13\frac{1}{2} \cdot \frac{2}{3}$

$\boxed{a_1 = 9}$

**(Total 3 marks)**

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**5.** The first and fourth terms of a geometric series are 18 and $-\frac{2}{3}$ respectively.

Find

(a) the sum of the first $n$ terms of the series; **(4)**

(b) the sum to infinity of the series. **(2)**

(a) $a_1 = 18$ $\quad a_4 = -\frac{2}{3}$

$a_4 = a_1 \cdot r^3$

$-\frac{2}{3} = 18 \cdot r^3$

$r^3 = -\frac{1}{27}$

$r = -\frac{1}{3}$

$S_n = \frac{18(1-(-\frac{1}{3})^n)}{1-(-\frac{1}{3})}$

$S_n = \frac{18(1-(-\frac{1}{3})^n)}{\frac{4}{3}}$

$\boxed{S_n = \frac{27(1-(-\frac{1}{3})^n)}{2}}$

(b) $S = \frac{a_1}{1-r}$

$S = \frac{18}{\frac{4}{3}}$

$\boxed{S = \frac{27}{2}}$

**(Total 6 marks)**

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**6.** Find the coefficient of $x^3$ in the binomial expansion of $\left(1-\frac{1}{2}x\right)^8$.

$\binom{8}{5}(1)^5\left(-\frac{1}{2}x\right)^3 = 56 \cdot 1 \cdot \left(-\frac{1}{8}x^3\right) = -7x^3$

$\binom{8}{5} = \binom{8}{3}$

$\boxed{-7}$

**(Total 4 marks)**

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**7.** Find the constant term in the binomial expansion of $\left(2x^2 - \frac{1}{x}\right)^9$.

$\binom{9}{3}(2x^2)^3\left(-\frac{1}{x}\right)^6 = 84 \cdot 8x^6 \cdot \left(\frac{1}{x^6}\right) = \boxed{672}$

**(Total 6 marks)**

**8.** Find the sum of all three-digit natural numbers that are not exactly divisible by 3.

ALL

100 − 999

$S_{900} = \frac{900}{2}(100 + 999)$

$S_{900} = 494{,}550$

DIV BY 3

102 − 999

$S_{300} = \frac{300}{2}(102 + 999)$

$S_{300} = 165{,}150$

NOT DIV BY 3

$494550 - 165150 = \boxed{329{,}400}$

**(Total 5 marks)**

**9.** How many four-digit numbers are there which contain at least one digit 3?

ALL 4-DIGIT #'s $\quad 9 \cdot 10 \cdot 10 \cdot 10 = 9{,}000$ ← CAN'T BEGIN W/ 0

No 3's $\quad 8 \cdot 9 \cdot 9 \cdot 9 = 5{,}832$

$\boxed{3{,}168}$

**(Total 3 marks)**

**10.** A local bridge club has 17 members, 10 females and 7 males. They have to elect 3 officers: president, deputy, and treasurer. In how many ways is this possible if: ORDER IMPORTANT, PERM.

(a) there are no restrictions? **(2)**

$17 \cdot 16 \cdot 15 = \boxed{4{,}080}$

(b) the president is male? **(2)**

$7 \cdot 16 \cdot 15 = \boxed{1680}$

(c) the president and deputy are the same gender? **(2)**

$\underbrace{7 \cdot 6}_{\text{MALE}} \cdot 15 + \underbrace{10 \cdot 9}_{\text{FEMALE}} \cdot 15 = 630 + 1350 = 1980$

**(Total 6 marks)**

**11.** Let $f$ and g be two functions. Given that $(f \circ g)(x) = \dfrac{x+1}{2}$ and $g(x) = 2x - 1$, find $f(x - 3)$.

$f(g(x)) = \frac{x+1}{2}$

$f(2x-1) = \frac{x+1}{2}$

$g^{-1}(x)$

$y = 2x - 1$

$x = 2y - 1$

$x + 1 = 2y$

$y = \frac{x+1}{2}$

$(f \circ g \circ g^{-1})(x) = \dfrac{\left(\frac{x+1}{2}\right) + 1}{2} = \frac{x+3}{4}$

$f(x) = \frac{x+3}{4}$

$f(x-3) = \frac{x-3+3}{4}$

$\boxed{f(x-3) = \frac{x}{4}}$

**(Total 6 marks)**

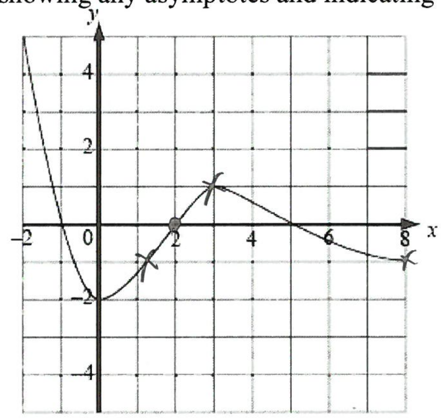
**12.** The function $f$ is defined by $f : x \mapsto x^3$.
Find an expression for $g(x)$ in terms of $x$ in each of the following cases

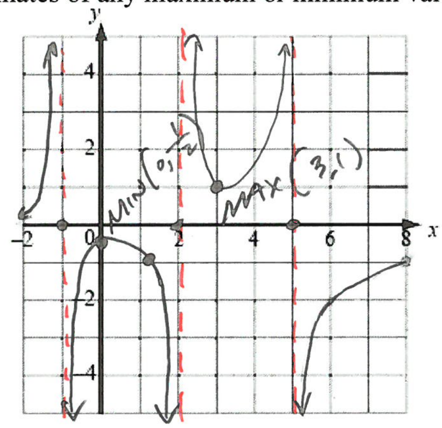
(a) $(f \circ g)(x) = x + 1$; $\quad g(x) = \sqrt[3]{x+1} \longrightarrow f(g(x)) = \left(\sqrt[3]{x+1}\right)^3 = x + 1$

(b) $(g \circ f)(x) = x + 1$. $\quad g(x) = \sqrt[3]{x} + 1 \longrightarrow g(f(x)) = \sqrt[3]{(x^3)} + 1 = x + 1$

**(Total 6 marks)**

**13.** The graph of $y = f(x)$ for $-2 \le x \le 8$ is shown. On the set of axes provided, sketch the graph of $y = \dfrac{1}{f(x)}$, clearly showing any asymptotes and indicating the coordinates of any maximum or minimum values.




**(Total 5 marks)**

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**14.** The functions $f$ and $g$ are defined below. Find the values of $x$ for which $(f \circ g)(x) \le (g \circ f)(x)$.

$$f(x) = 2x - 1 \qquad g(x) = \frac{x}{x+1}, x \ne -1.$$

$$f(g(x)) = 2\left(\frac{x}{x+1}\right) - 1 = \frac{2x}{x+1} - \frac{x+1}{x+1} = \boxed{\frac{x-1}{x+1}}$$

$$g(f(x)) = \frac{2x-1}{(2x-1)+1} = \boxed{\frac{2x-1}{2x}}$$

$$\frac{x-1}{x+1} \le \frac{2x-1}{2x}$$ CRITICAL PTS: $x = -1, 0$ (DENOM $= 0$)

SOLVE =

$$\frac{x-1}{x+1} \cancel{=} \frac{2x-1}{2x}$$

$$2x(x-1) = (x+1)(2x-1)$$

$$\cancel{2x^2} - 2x = \cancel{2x^2} + 2x - x - 1$$

$$-3x = -1$$

$x = \frac{1}{3}$ CRITICAL PT.

TEST VALUES $\left(\frac{x-1}{x+1} \le \frac{2x-1}{2x}\right)$

✗ $-2$, $-1$, ✓ $-\frac{1}{2}$, $0$, ✗ $\frac{1}{4}$, $\frac{1}{3}$, ✓ $1$

$3 \not\le \frac{5}{4}$ $\quad -3 \le 1$ $\quad -\frac{3}{5} \not\le -1$ $\quad 0 \le \frac{1}{2}$

SOLUTION

$$\boxed{(-1, 0) \cup [1/3, \infty) \quad \text{or} \quad -1 < x < 0 \text{ AND } x \ge 1/3}$$

**(Total 6 marks)**

**15.** Let $f(x) = \dfrac{4}{x+2}, x \neq -2$ and $g(x) = x - 1$.

If $h = g \circ f$, find

(a) $h(x)$; $g(f(x)) = \left(\dfrac{4}{x+2}\right) - 1 = \dfrac{4-(x+2)}{x+2} = \boxed{\dfrac{2-x}{x+2}}$ **(2)**

(b) $h^{-1}(x)$, where $h^{-1}$ is the inverse of $h$. **(4)**

$y = \dfrac{4}{x+2} - 1$

$x = \dfrac{4}{y+2} - 1$

$x + 1 = \dfrac{4}{y+2}$

$y+2 = \dfrac{4}{x+1}$

$y = \dfrac{4}{x+1} - 2$

$y = \dfrac{4-2(x+1)}{x+1} = \boxed{\dfrac{2-2x}{x+1}}$

**(Total 6 marks)**

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**16.** Given functions $f(x) = 2x + 1$ and $g(x) = x^3$, find the function $(f^{-1} \circ g)^{-1}$.

$y = 2x+1$

$x = 2y+1$

$2y = x-1$

$y = \dfrac{x-1}{2}$

$f^{-1}(g(x)) = \dfrac{x^3-1}{2}$

$y = \dfrac{x^3-1}{2}$

$x = \dfrac{y^3-1}{2}$

$2x = y^3 - 1$

$y^3 = 2x+1$

$\boxed{y = \sqrt[3]{2x+1}}$

**(Total 4 marks)**

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**17.** The function $f$ is defined for $x \leq 0$ by $f(x) = \dfrac{x^2-1}{x^2+1}$. Find an expression for $f^{-1}(x)$.

$y = \dfrac{x^2-1}{x^2+1}$

$x = \dfrac{y^2-1}{y^2+1}$

$x(y^2+1) = y^2 - 1$

$xy^2 + x = y^2 - 1$

$xy^2 - y^2 = -x - 1$

$\dfrac{y^2(x-1)}{x-1} = \dfrac{-x-1}{x-1}$

$y^2 = \dfrac{-x-1}{x-1}$

$y^2 = \dfrac{x+1}{1-x}$

$y = \pm\sqrt{\dfrac{x+1}{1-x}}$

$\boxed{y = -\sqrt{\dfrac{x+1}{1-x}}}$

**(Total 6 marks)**

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**18.** Shown below are the graphs of $y = f(x)$ and $y = g(x)$.
If $(f \circ g)(x) = 3$, find all possible values of $x$.

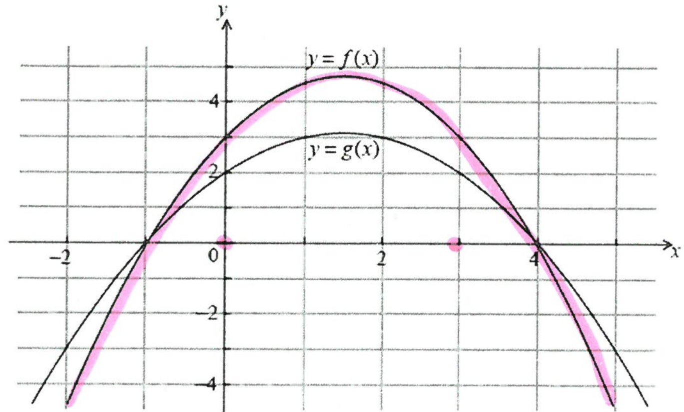
$f(g(x)) = 3$

$f(x) = 3$ @ 0 AND 3

$g(x) = 0$ @ $-1$ AND 4

$g(x) = 3$ @ 1 AND 2

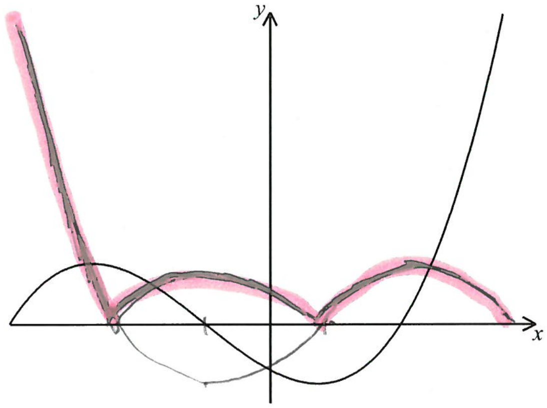
$\boxed{x = -1, 1, 2, 4}$




**(Total 4 marks)**

**19.** Each of the diagrams below shows the graph of a function $f$. Sketch on the given axes the graph of

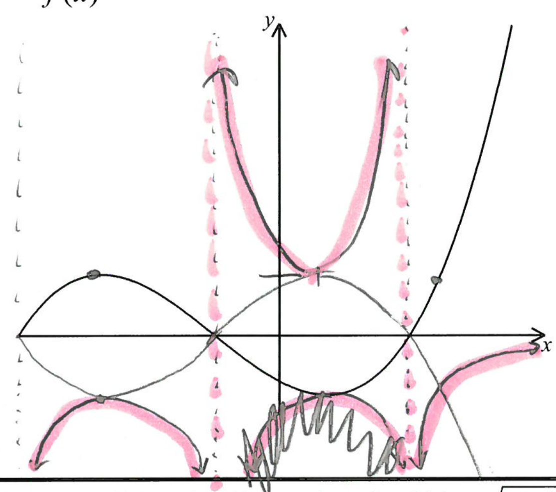
(a) $|f(-x)|$;




$f(-x) \rightarrow$ FLIPS OVER Y-AXIS

$|f(-x)| \rightarrow$ BELOW X-AXIS FLIPS UP

(b) $\dfrac{1}{-f(x)}$.




$-f(x) \longrightarrow$ FLIPS OVER X-AXIS

$\dfrac{1}{-f(x)} \longrightarrow$ BIG $\rightarrow$ SMALL, SMALL $\rightarrow$ BIG) RECIPROCALS

ZERO $\rightarrow$ ASYMP

$y=1 \rightarrow$ STAYS SAME

**(Total 6 marks)**

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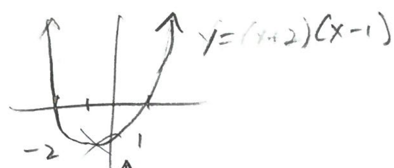
**20.** The functions $f(x)$ and $g(x)$ are given by $f(x) = \sqrt{x-2}$ and $g(x) = x^2 + x$. The function $(f \circ g)(x)$ is defined for $x \in \mathbb{R}$, **except** for the interval $]\, a, b\, [$.

(a) Calculate the value of $a$ and of $b$.

(b) Find the range of $f \circ g$.

$f(g(x)) = \sqrt{x^2+x-2} = \sqrt{(x+2)(x-1)}$



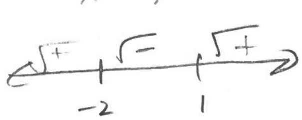

(a) SEE GRAPH

(b) RANGE: $y \geq 0$

or

$(x+2)(x-1)=0$

$x=-2, 1$




**(Total 6 marks)**

**21.** A sum of \$5 000 is invested at a compound interest rate of 6.3% annually.

(a) Write down an expression for the value of the investment after $n$ full years.

(b) What will be the value of the investment at the end of five years?

(c) The value of the investment will exceed \$10 000 after $n$ full years.

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of $n$.

(a) $\boxed{P = 5{,}000\,(1.063)^n}$

(b) $P = 5000\,(1.063)^5$

$\boxed{P = \$6786.35}$

(c) $\boxed{5000\,(1.063)^n > 1000}$

$1.063^n > 2$

$\ln 1.063^n > \ln 2$

$n > \dfrac{\ln 2}{\ln 1.063}$

$n > 11.34538\ldots$

$\boxed{n = 12}$

**(Total 6 marks)**

**22.** Solve the equation $\log_3(x + 17) - 2 = \log_3 2x$.

$\log_3(x+17) - 2\log_3 3 = \log_3 2x$

$\log_3\left(\dfrac{x+17}{3^2}\right) = \log_3 2x$

$\dfrac{x+17}{9} = 2x$

$x + 17 = 18x$

$\boxed{x = 1}$

**(Total 5 marks)**

**23.** Given that $4 \ln 2 - 3\ln 4 = -\ln k$, find the value of $k$.

$\ln 2^4 - \ln 4^3 = -1 \cdot \ln k$

$\ln \dfrac{16}{64} = \ln k^{-1}$

$k^{-1} = \dfrac{1}{4}$

$\boxed{k = 4}$

**(Total 5 marks)**

**24.** Solve the equation $2^{2x+2} - 10 \times 2^x + 4 = 0,\ x \in \mathbb{R}$.

$2^{2x} \cdot 2^2 - 10 \cdot 2^x + 4 = 0$

$4 \cdot (2^x)^2 - 10(2^x) + 4 = 0$

$y = 2^x$

$\dfrac{4y^2 - 10y + 4 = 0}{2}$

$2y^2 - 5y + 2 = 0$

$(2y-1)(y-2) = 0$

$y = \frac{1}{2}, 2$

$2^x = \frac{1}{2} \qquad 2^x = 2$

$x = -1 \qquad x = 1$

$\boxed{x = \pm 1}$

**(Total 6 marks)**

**25.** Find the **exact** value of $x$ satisfying the equation

$$(3^x)(4^{2x+1}) = 6^{x+2}.$$

Give your answer in the form $\frac{\ln a}{\ln b}$ where $a, b \in \mathbb{Z}$.

$(3^x)(4^{2x+1}) = 6^{x+2}$

$(3^x)((2^2)^{2x+1}) = (2\cdot 3)^{x+2}$

$3^x \cdot 2^{4x+2} = 2^{x+2} \cdot 3^{x+2}$

$\frac{2^{4x+2}}{2^{x+2}} = \frac{3^{x+2}}{3^x}$

$2^{3x} = 3^2$

$\ln 2^{3x} = \ln 3^2$

$\frac{3x \cdot \ln 2}{3\ln 2} = \frac{2 \cdot \ln 3}{3 \ln 2}$

$x = \frac{2\cdot \ln 3}{3 \ln 2}$

$x = \frac{\ln 3^2}{\ln 2^3}$

$\boxed{x = \frac{\ln 9}{\ln 8}}$

**(Total 6 marks)**

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**26.** Solve the equation $9 \log_5 x = 25 \log_x 5$, expressing your answers in the form $5^{\frac{p}{q}}$, where $p, q \in \mathbb{Z}$.

CHANGE OF BASE

$\frac{9\cdot \log x}{\log 5} = \frac{25 \cdot \log 5}{\log x}$

$9\cdot(\log x)^2 = 25\cdot(\log 5)^2$

$\left(\frac{\log x}{\log 5}\right)^2 = \frac{25}{9}$

$\frac{\log x}{\log 5} = \frac{5}{3}$

$\log_5 x = \frac{5}{3}$

$\boxed{x = 5^{5/3}}$

$\left(\text{BASE}^{\text{ANSWER}} = \text{\# TAKING LOG OF}\right)$

**(Total 5 marks)**

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**27.** Solve the following equations.

(a) $\ln(x + 2) = 3$.

$e \qquad e$

$x+2 = e^3$

$\boxed{x = e^3 - 2}$

(b) $\ln 10^{2x} = \ln 500$.

$2$

$2x \cdot \ln 10 = \ln 500$

$2x = \frac{\ln 500}{\ln 10}$

$\boxed{x = \frac{\ln 500}{2\cdot \ln 10}}$

$1.349485\ldots$

**(Total 5 marks)**

**28.** (a) Solve the equation $2(4^x) + 4^{-x} = 3$.

$\left(2\cdot 4^x + \frac{1}{4^x} = 3\right)\cdot 4^x$

**(5)**

$2(4^x)^2 + 1 = 3\cdot 4^x$

$y = 4^x$ $\quad 2(4^x)^2 - 3(4^x) + 1 = 0$

$2y^2 - 3y + 1 = 0$

$(2y-1)(y-1) = 0 \quad y = \frac{1}{2}, 1$

$4^x = \frac{1}{2} \quad 4^x = 1$

$\boxed{x = -\frac{1}{2} \quad x = 0}$

(b) (i) Solve the equation $a^x = e^{2x+1}$ where $a > 0$, giving your answer for $x$ in terms of $a$. $\quad x = a$

(ii) For what value of $a$ does the equation have no solution?

**(6)**

$\ln a^x = \ln e^{2x+1}$

$x\cdot \ln a = 2x + 1$

$x\cdot \ln a - 2x = 1$

$x(\ln a - 2) = 1$

$\boxed{x = \frac{1}{\ln a - 2}}$

**(Total 11 marks)**

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**29.** Solve the equation $4^{x-1} = 2^x + 8$.

$(2^2)^{x-1} = 2^x + 8$

$2^{2x-2} = 2^x + 8$

$2^{2x}\cdot 2^{-2} = 2^x + 8$

$\frac{1}{4}(2^x)^2 = 2^x + 8$

$y = 2^x$

$\frac{1}{4}y^2 = y + 8$

$\left(\frac{1}{4}y^2 - y - 8 = 0\right)\cdot 4$

$y^2 - 4y - 32 = 0$

$(y-8)(y+4) = 0$

$y = 8, -4$

$2^x = 8 \quad \cancel{2^x = -4}$

$\boxed{x = 3}$ NO SOLUTION

**(Total 5 marks)**

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**30.** The solution of $2^{2x+3} = 2^{x+1} + 3$ can be expressed in the form $a + \log_2 b$ where $a, b \in \mathbb{Z}$. Find the value of $a$ and of $b$.

$2^{2x}\cdot 2^3 = 2^x\cdot 2^1 + 3$

$8\cdot(2^x)^2 = 2\cdot(2^x) + 3$

$y = 2^x$

$8y^2 = 2y + 3$

$8y^2 - 2y - 3 = 0$

$(4y-3)(2y+1) = 0$

$y = \frac{3}{4}, -\frac{1}{2}$

$2^x = \frac{3}{4} \qquad 2^x = -\frac{1}{2}$

NO SOLUTION

$\log_2 2^x = \log_2 \frac{3}{4}$

$x = \log_2 \frac{3}{4}$

$x = \log_2 3 - \log_2 4$

$x = \log_2 3 - \log_2 2^2$

$x = \log_2 3 - 2$

$\boxed{x = -2 + \log_2 3}$

$a = -2$

$b = 3$

**(Total 6 marks)**

**31.** (a) Find the solution of the equation, expressing your answer in terms of ln 2.

$$\ln 2^{4x-1} = \ln 8^{x+5} + \log_2 16^{1-2x},$$

$\ln 2^{4x-1} = \ln 2^{3x+15} + \log_2 2^{4-8x}$

$[(4x-1)-(3x+15)]\cdot \ln 2 = 4-8x$

$(x-16)\cdot \ln 2 = 4-8x$

$x\cdot \ln 2 + 8x = 4 + 16\cdot \ln 2$

$x\cdot(\ln 2 + 8) = 4 + 16\cdot \ln 2$

$\boxed{x = \dfrac{4+16\cdot\ln 2}{8+\ln 2}}$

**(4)**

(b) Using this value of $x$, find the value of $a$ for which $\log_a x = 2$, giving your answer to three decimal places.

**(2)**

$\log_a x = 2$

$a^2 = x$

$a = +\sqrt{x}$

IGNORE "−"

$a = \sqrt{\dfrac{4+16\cdot\ln 2}{8+\ln 2}}$

$\boxed{a \approx 1.318}$

**(Total 6 marks)**

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**32.** The mass $m$ kg of a radio-active substance at time $t$ hours is given by $m = 4e^{-0.2t}$. If the mass of the substance is reduced by one quarter, how long does this take (to the nearest tenth of an hour)?

$m = 4e^{-0.2t}$

INITIAL MASS

$3 = 4e^{-0.2t}$

$\frac{3}{4} = e^{-0.2t}$

$\ln\frac{3}{4} = -0.2t$

$t = \dfrac{\ln\frac{3}{4}}{-0.2}$

$t = 1.4384103622\ldots$

$\boxed{t \approx 1.4 \text{ hrs}}$

**(Total 4 marks)**

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**33.** Solve, for $x$, the equation $\log_2(5x^2 - x - 2) = 2 + 2\log_2 x$.

$\log_2(5x^2-x-2) = 2\log_2 2 + \log_2 x^2$

$\log_2(5x^2-x-2) = \log_2(4x^2)$

$5x^2-x-2 = 4x^2$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x = 2, \cancel{-1}$

CANNOT LOG NEGATIVE #

$\boxed{x=2}$

ALT. METHOD

$\log_2(5x^2-x-2) - \log_2(x^2) = 2$

$\log_2\left(\dfrac{5x^2-x-2}{x^2}\right) = 2$

BASE$^{\text{ANSWER}}$ = # TAKING LOG OF

$2^2 = \dfrac{5x^2-x-2}{x^2}$

$4x^2 = 5x^2-x-2$

$\boxed{x=2}$

**(Total 5 marks)**

$x = ?$

**34.** Find the **exact** value of $x$ in each of the following equations.

(a) $5^{x+1} = 625$

$5^{x+1} = 5^4$

$x+1=4$

$\boxed{x=3}$

(b) $\log_{\mathbf{a}}(3x+5) = 2$

$a^2 = 3x+5$

$3x = a^2 - 5$

$\boxed{x = \frac{a^2-5}{3}}$

**(Total 4 marks)**

---

**35.** Let $f(x) = \log_a x$, $x > 0$.

(a) Write down the value of

(i) $f(a)$; $\log_a a = \boxed{1}$

(ii) $f(1)$; $\log_a 1 = \boxed{0}$

(iii) $f(a^4)$. $\log_a a^4 = \boxed{4}$

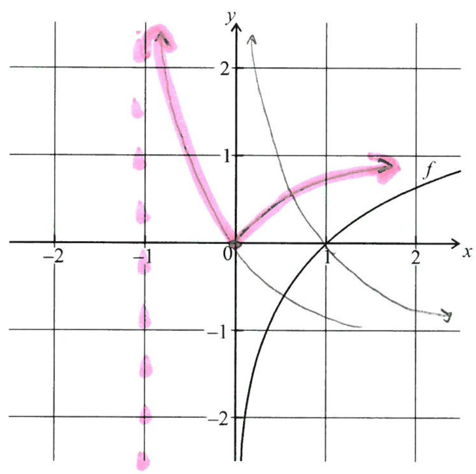
(b) The diagram below shows part of the graph of $f$.

On the same diagram, sketch the graph of $|-f(x+1)|$.

$-f(x) \to$ FLIP OVER Y-AXIS

$f(x+1) \to$ SHIFT 1 ←

$|f(x)| =$ FLIP ANYTHING BELOW X-AXIS




**(Total 6 marks)**

---

**36.** Solve $\log_{16} \sqrt[3]{100 - x^2} = \frac{1}{2}$.

$16^{1/2} = \sqrt[3]{100-x^2}$

$4 = \sqrt[3]{100-x^2}$

$64 = 100 - x^2$

$x^2 = 36$

$\boxed{x = \pm 6}$

**(Total 4 marks)**

---

**37.** Write $\ln(x^2 - 1) - 2\ln(x+1) + \ln(x^2 + x)$ as a single logarithm, in its simplest form.

$\ln((x+1)(x-1)) - \ln(x+1)^2 + \ln(x(x+1))$

$\ln \frac{\cancel{(x+1)}(x-1)\, x\cancel{(x+1)}}{\cancel{(x+1)}\cancel{(x+1)}}$

$\ln(x(x-1))$

$\boxed{\ln(x^2 - x)}$

**(Total 5 marks)**

**38.** Solve the equations

$$\ln \frac{x}{y} = 1$$

$$\ln x^3 + \ln y^2 = 5.$$

$(\ln x - \ln y = 1)\cdot 2$

$3\ln x + 2\ln y = 5$

$2\ln x - 2\ln y = 2$

$3\ln x + 2\ln y = 5$

$5\ln x = 7$

$\ln x = \frac{7}{5}$

$\frac{7}{5} - \ln y = 1$

$\ln y = \frac{2}{5}$

$\ln x = \frac{7}{5}$ $\quad$ $\ln y = \frac{2}{5}$

$\boxed{x = e^{7/5} \quad y = e^{2/5}}$

**(Total 5 marks)**

**39.** Solve $2(5^{x+1}) = 1 + \frac{3}{5^x}$, giving the answer in the form $a + \log_5 b$, where $a, b \in \mathbb{Z}$.

$2\cdot 5^x \cdot 5^1 = 1 + \frac{3}{5^x}$

$\left(10\cdot 5^x - 1 - \frac{3}{5^x} = 0\right)\cdot 5^x$

$10(5^x)^2 - 5^x - 3 = 0$

$y = 5^x$

$10y^2 - y - 3 = 0$

$(5y - 3)(2y + 1) = 0$

$y = \frac{3}{5}$ $\quad$ $y = -\frac{1}{2}$

NO SOLUTION

$5^x = \frac{3}{5}$

$\log_5 5^x = \log_5 \frac{3}{5}$

$x = \log_5 3 - \log_5 5$

$x = \log_5 3 - 1$

$\boxed{x = -1 + \log_5 3}$ ✓

$a = -1$ $\quad$ $b = 3$

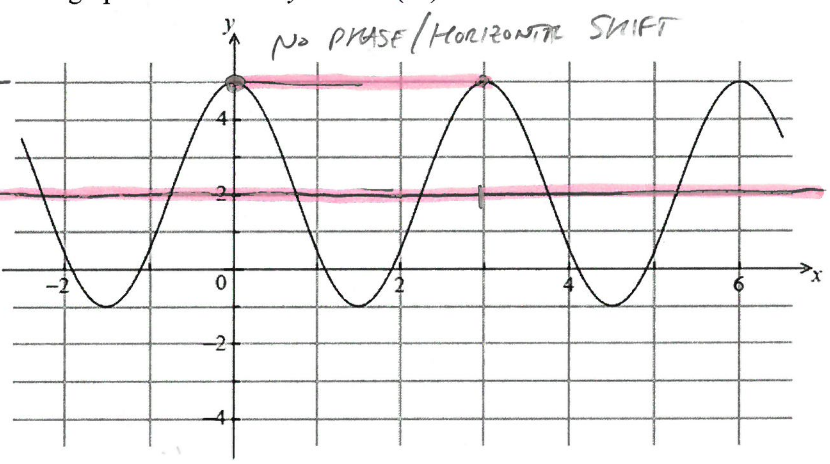
**(Total 6 marks)**

**40.** The graph below shows $y = a\cos(bx) + c$.

NO PHASE / HORIZONTAL SHIFT

$a = 3$

$c = +2$




PERIOD $= \frac{2\pi}{b}$

$3 = \frac{2\pi}{b}$

$b = \frac{2\pi}{3}$

Find the value of $a$, the value of $b$ and the value of $c$.

$\boxed{a = 3, \quad b = \frac{2\pi}{3}, \quad c = 2}$

**(Total 4 marks)**

**41.** The depth, $h(t)$ meters, of water at the entrance to a harbor at $t$ hours after midnight on a particular day is given by:

$$h(t) = 8 + 4\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 24.$$

VERT a

(a) Find the maximum depth and the minimum depth of the water. **(3)**

VERT SHIFT = 8↑

AMPLITUDE = 4

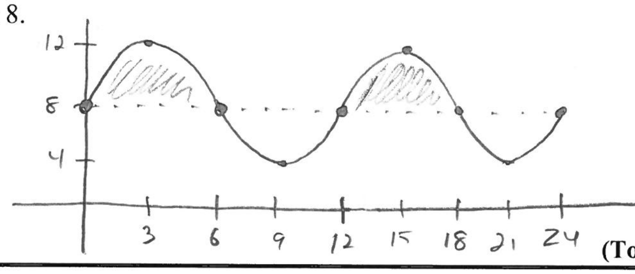
$\boxed{\text{MAX} = 8+4 = 12\text{ m} \quad \text{MIN} = 8-4 = 4\text{ m}}$

(b) Find the values of $t$ for which $h(t) \ge 8$. **(3)**

$8 + 4\sin\left(\frac{\pi}{6}\cdot t\right) \ge 8$

$\sin\left(\frac{\pi}{6}\cdot t\right) \ge 0$

$\boxed{0 \le t \le 6,\ 12 \le t \le 18,\ t = 24}$




**(Total 6 marks)**

**42.** Solve $\sin 2x = \sqrt{2}\cos x, 0 \le x \le \pi$.

$2\sin x\cos x - \sqrt{2}\cos x = 0$ (DO NOT DIVIDE BY COS x !!)

$\cos x(2\sin x - \sqrt{2}) = 0$

$\cos x = 0 \qquad \sin x = \frac{\sqrt{2}}{2}$

$x = \frac{\pi}{2} \qquad x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}}$

— C +
S
—

**(Total 6 marks)**

**43.** The angle $\theta$ satisfies the equation $2\tan^2\theta - 5\sec\theta - 10 = 0$, where $\theta$ is in the second quadrant. Find the value of $\sec\theta$.

PYTH. IDENTITY $1 + \tan^2\theta = \sec^2\theta$

$2(\sec^2\theta - 1) - 5\sec\theta - 10 = 0$

$2\sec^2\theta - 5\sec\theta - 12 = 0$

$(2\sec\theta + 3)(\sec\theta - 4) = 0$

$\sec\theta = -\frac{3}{2}, 4$

$\boxed{\sec\theta = -\frac{3}{2}}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cos\theta$ in Q2 is —

**(Total 6 marks)**

**44.** Solve $2\sin x = \tan x$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$2\sin x = \frac{\sin x}{\cos x}$ (DO NOT DIVIDE BY SIN x)

$2\sin x - \frac{\sin x}{\cos x} = 0$

$\sin x\left(2 - \frac{1}{\cos x}\right) = 0$

$\sin x = 0 \qquad \frac{1}{\cos x} = 2$

$\cos x = \frac{1}{2}$

$x = 0 \qquad x = \frac{\pi}{3}, -\frac{\pi}{3}$

$\boxed{x = 0, \pm\frac{\pi}{3}}$

**(Total 3 marks)**

**45.** Given that $\tan 2\theta = \frac{3}{4}$, find the possible values of $\boxed{\tan \theta}$.

DOUBLE ∡ / SEE PACKET

$$\frac{2\tan\theta}{1-\tan^2\theta} \times \frac{3}{4}$$

$$8\tan\theta = 3 - 3\tan^2\theta$$

$$3\tan^2\theta + 8\tan\theta - 3 = 0$$

$$(3\tan\theta - 1)(\tan\theta + 3) = 0$$

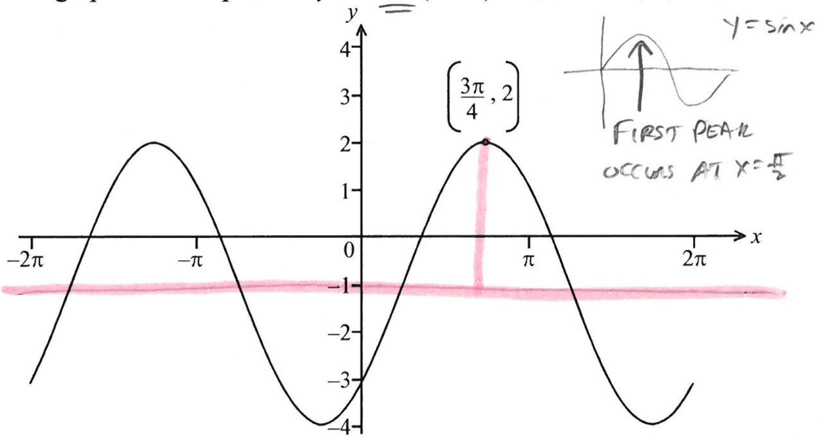
$$\tan\theta = \frac{1}{3} \qquad \tan\theta = -3$$

$$\boxed{\tan\theta = \frac{1}{3}, -3}$$

**(Total 5 marks)**

---

**46.** The graph below represents $y = a \sin (x + b) + c$, where $a$, $b$, and $c$ are constants.




PERIOD → UNCHANGED

$a = 3$

$b = -\frac{\pi}{4}$

$c = -1$

$$y = 3\sin\left(x + \left(-\frac{\pi}{4}\right)\right) - 1$$

TEST POINT $\left(\frac{3\pi}{4}, 2\right)$

Find values for $a$, $b$ and $c$.

$$y = 3\cdot\sin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) - 1$$

$$y = 3\cdot\sin\left(\frac{\pi}{2}\right) - 1$$

$$y = 3 - 1$$

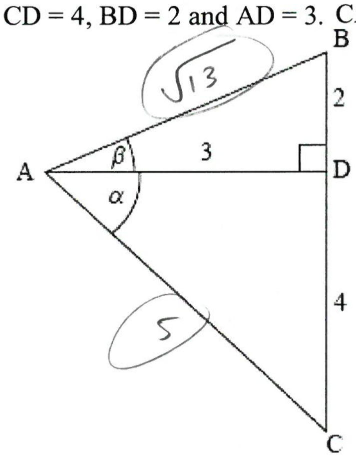
$y = 2$ ✓

**(Total 6 marks)**

---

**47.** In the diagram below, AD is perpendicular to BC.

CD = 4, BD = 2 and AD = 3. $C\hat{A}D = \alpha$ and $B\hat{A}D = \beta$.




$\sin\beta = \frac{2}{\sqrt{13}}$

$\cos\beta = \frac{3}{\sqrt{13}}$

$\sin\alpha = \frac{4}{5}$

$\cos\alpha = \frac{3}{5}$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \left(\frac{3}{5}\right)\left(\frac{3}{\sqrt{13}}\right) + \left(\frac{4}{5}\right)\left(\frac{2}{\sqrt{13}}\right)$$

$$= \frac{9}{5\sqrt{13}} + \frac{8}{5\sqrt{13}}$$

$$= \frac{17}{5\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}}$$

$$\boxed{= \frac{17\sqrt{13}}{65}}$$

Find $\boxed{\text{the exact v}}$alue of $\cos(\alpha - \beta)$.

SUM/DIFFERENCE/IN PACKET

NO DECIMALS... MUST BE IN THIS FORM!!

**(Total 6 marks)**

START FROM MORE COMPLICATED SIDE

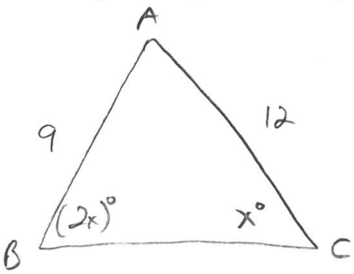
**48.** Verify the identity $\boxed{\dfrac{\cot^2\theta}{1+\csc\theta}} = \dfrac{1-\sin\theta}{\sin\theta}$.

$$\frac{\csc^2\theta - 1}{1+\csc\theta} = \frac{(\csc\theta-1)(\csc\theta+1)}{\csc\theta+1} = \csc\theta - 1 = \frac{1}{\sin\theta} - 1 = \frac{1}{\sin\theta} - \frac{\sin\theta}{\sin\theta}$$

$$= \boxed{\frac{1-\sin\theta}{\sin\theta}} \checkmark$$

**(Total 6 marks)**

**49.** In triangle ABC, AB = 9 cm, AC = 12 cm, and $\angle B$ is twice the size of $\angle C$. Find the cosine of $\angle C$.




LAW OF SINES

$$\frac{\sin 2x}{12} = \frac{\sin x}{9}$$

$$9 \cdot 2\sin x\cos x = 12 \cdot \sin x$$

$$18\sin x\cos x - 12\sin x = 0$$

$$6\sin x(3\cos x - 2) = 0$$

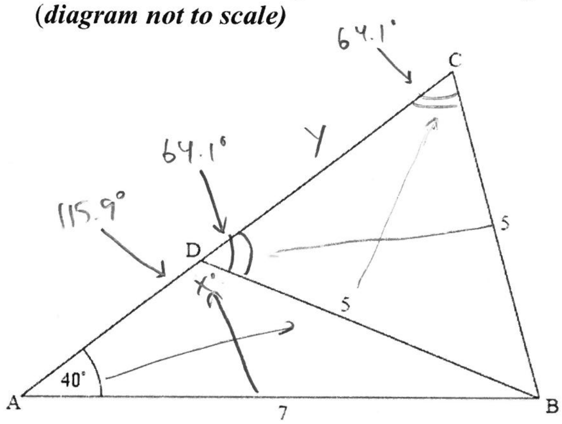
$$6\sin x = 0 \qquad \cos x = \frac{2}{3}$$

$$\boxed{\cos\angle C = \frac{2}{3}}$$

NO NEED TO FIND THE ACTUAL VALUE OF X

**(Total 5 marks)**

**50.** Given ΔABC, with lengths shown in the diagram below, find the length of the line segment [CD].
***(diagram not to scale)***




$$\frac{\sin 40}{5} = \frac{\sin x}{7}$$

$$x = \sin^{-1}\left(\frac{7\cdot\sin 40}{5}\right)$$

$$x = 64.1^\circ \text{ or } 115.9^\circ$$

MUST BE SINCE △BDC ISOS. AND ∠C CANNOT BE 115.9

$$\angle B = 180 - 2\cdot 64.1$$

$$\angle B = 51.7^\circ$$

$$\frac{\sin 64.1^\circ}{5} = \frac{\sin 51.7}{Y}$$

$$Y = \frac{5\cdot\sin 51.7}{\sin 64.1}$$

$$\boxed{Y \approx 4.36}$$

**(Total 5 marks)**

**51.** A fair six-sided die, with sides numbered 1, 1, 2, 3, 4, 5 is thrown. Find the mean and variance of the score.

$$\text{MEAN} = \frac{1+1+2+3+4+5}{6} = \frac{16}{6} = \boxed{\frac{8}{3}}$$

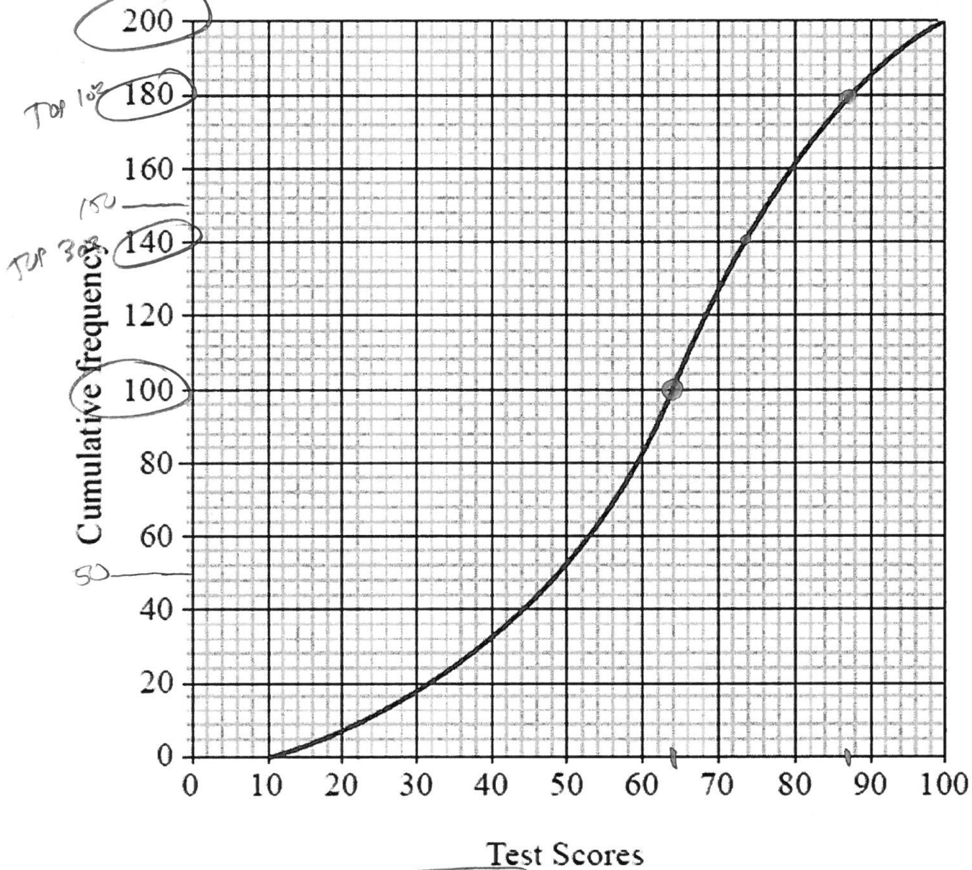
$$\text{VARIANCE} = \frac{1^2+1^2+2^2+3^2+4^2+5^2}{6} - \left(\frac{8}{3}\right)^2 = \boxed{\frac{20}{9}}$$

$$\text{VAR} = \frac{\text{SUM OF SQUARES}}{\#\text{ ITEMS}} - \text{MEAN}^2$$

**(Total 6 marks)**

(IN PACKET BUT IN Σ FORM)

**52.** The test scores of a group of students are shown on the cumulative frequency graph below.




(a) Estimate the median test score. $\boxed{64}$ **(1)**

(b) The top 10 % of students receive a grade A and the next best 20 % of students receive a grade B.

Estimate TOP 20 TOP 60

(i) the minimum score required to obtain a grade A; $\boxed{87}$ $\pm 1$

(ii) the minimum score required to obtain a grade B. $\boxed{74}$ $\pm 1$

**(4)**

**(Total 5 marks)**

**53.** In a sample of 50 boxes of light bulbs, the number of defective light bulbs per box is shown below.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Number of defective light bulbs per box | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| Number of boxes | 7 | 3 | 15 | 11 | 6 | 5 | 3 |
| CUMULATIVE FREQUENCY | 7 | 10 | 25 | 36 | 42 | 47 | 50 |

25/26

(a) Calculate the median number of defective light bulbs per box.

$\frac{2+3}{2} = \boxed{2.5}$

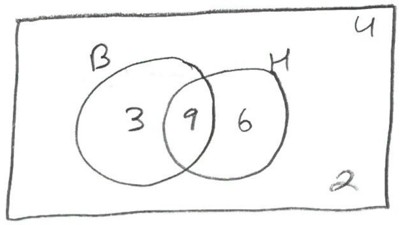
(b) Calculate the mean number of defective light bulbs per box.

$\frac{0 \cdot 7 + 1 \cdot 3 + 2 \cdot 15 + 3 \cdot 11 + 4 \cdot 6 + 5 \cdot 5 + 6 \cdot 3}{50} = \boxed{2.8}$

**(Total 6 marks)**

**54.** In a class of 20 students, 12 study Biology, 15 study History and 2 students study neither Biology nor History.

(a) Illustrate this information on a Venn diagram. **(2)**




X = # TAKING BOTH

$12 + 15 - x + 2 = 20$

$x = 9$

(b) Find the probability that a randomly selected student from this class is studying both Biology and History. **(1)**

$P(B \cap H) = \frac{9}{20} = \boxed{0.45}$

(c) Given that a randomly selected student studies Biology, find the probability that this student also studies History. **(1)**

$P(H|B) = \frac{P(H \cap B)}{P(B)} = \frac{9/20}{12/20} = \boxed{\frac{3}{4}}$

**(Total 4 marks)**

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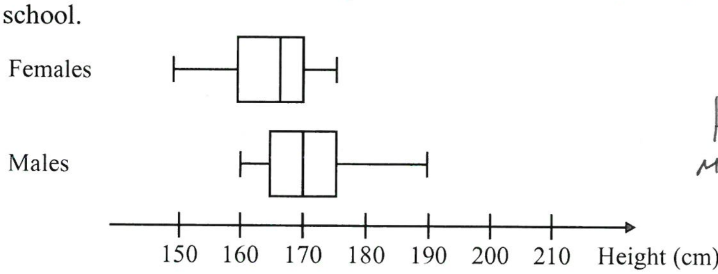
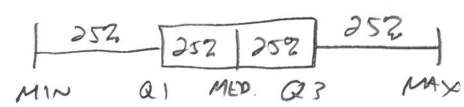
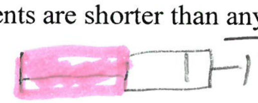
**55.** The box-and-whisker plots shown represent the heights of female students and the heights of male students at a certain school.




(a) What percentage of female students are shorter than any male students?

25%



(b) What percentage of male students are shorter than some female students?

75%

(c) From the diagram, **estimate** the mean height of the male students.

TOP HEAVY, > 170 (171-173) NOT 170!

**(Total 3 marks)**

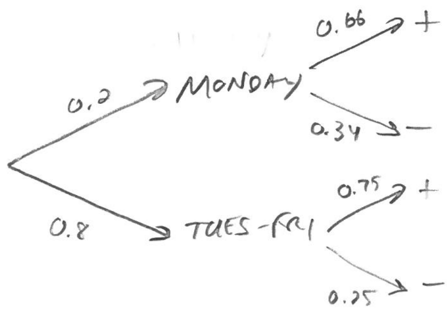
**56.** Robert travels to work by train every weekday from Monday to Friday. The probability that he catches the 08.00 train on Monday is 0.66. The probability that he catches the 08.00 train on any other weekday is 0.75. A weekday is chosen at random.

(a) Find the probability that he catches the train on that day.

(b) Given that he catches the 08.00 train on that day, find the probability that the chosen day is Monday.

+ = CATCHES TRAIN

− = MISSES TRAIN

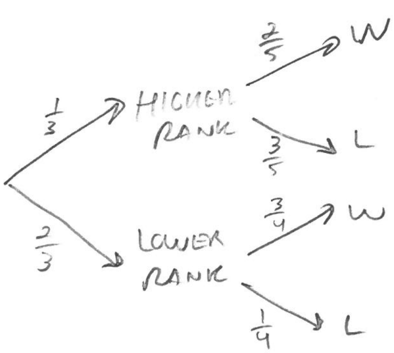



(a) $P(+) = 0.2 \cdot 0.66 + 0.8 \cdot 0.75 = \boxed{0.732}$

(b) $P(M|+) = \dfrac{P(M \cap +)}{P(+)} = \dfrac{0.2 \cdot 0.66}{0.732} = \boxed{0.180}$

**(Total 6 marks)**

**57.** The local Football Association consists of ten teams. Team *A* has a 40% chance of winning any game against a higher-ranked team, and a 75% chance of winning any game against a lower-ranked team. If *A* is currently in fourth position, find the probability that *A* wins its next game.




1 2 3 $\boxed{4}$ 5 6 7 8 9 10

3 TEAMS 6 TEAMS

$P(w) = \frac{1}{3} \cdot \frac{2}{5} + \frac{2}{3} \cdot \frac{3}{4}$

$P(w) = \frac{2}{15} + \frac{1}{2}$

$\boxed{P(w) = \frac{19}{30} \text{ OR } 0.6\overline{3}}$

**(Total 4 marks)**

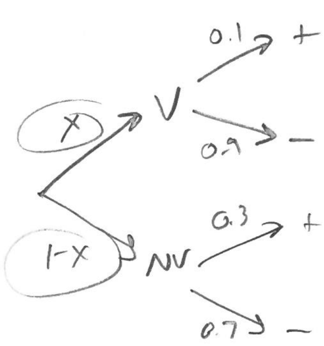
**58.** An influenza virus is spreading through a city. A vaccination is available to protect against the virus. If a person has had the vaccination, the probability of catching the virus is 0.1; without the vaccination, the probability is 0.3. The probability of a randomly selected person catching the virus is 0.22.

(a) Find the percentage of the population that has been vaccinated. **(3)**

(b) A randomly chosen person catches the virus. Find the probability that this person has been vaccinated. **(2)**

V = VACCINATED  + = CATCH VIRUS

NV = NOT VACC.  − = DO NOT CATCH




$P(+) = 0.22$

(a) $x \cdot 0.1 + (1-x) \cdot 0.3 = 0.22$

$0.1x + 0.3 - 0.3x = 0.22$

$0.2x = 0.08$

$x = 0.4$

$\boxed{P(V) = 0.4}$

(b) $P(V|+) = \dfrac{P(V \cap +)}{P(+)}$

$P(V|+) = \dfrac{0.4 \cdot 0.1}{0.22}$

$\boxed{P(V|+) = 0.\overline{18}}$

**(Total 5 marks)**

NO ROOM FOR 59–63! USE SEPARATE PAPER IF NECESSARY.

**59.** Use mathematical induction to prove that $1+2+3+...+n=\dfrac{n(n+1)}{2}$ for $n \in \mathbb{Z}^+$.

① PROVE $P(1)$ TRUE

$1 = \frac{1(1+1)}{2}$

$1 = 1$ ✓

② ASSUME $P(k)$ TRUE

$1+2+...+k = \frac{k(k+1)}{2}$

③ PROVE $P(k+1)$ TRUE

$1+2+...+k+(k+1) = \frac{(k+1)(k+2)}{2}$

FROM PART 2:

$1+2+...+k = \frac{k(k+1)}{2}$

$1+2+...+k+(k+1) = \frac{k(k+1)}{2}+k+1$

$'' = \frac{k^2+k+2k+2}{2}$

$'' = \frac{k^2+3k+2}{2}$

$'' = \frac{(k+1)(k+2)}{2}$ ✓

**60.** Use mathematical induction to prove that $1+3+5+7+...+(2n-1)=n^2$ for $n \in \mathbb{Z}^+$.

① $2(1)-1 = 1^2$

$1 = 1$ ✓

② $1+3+...+(2k-1) = k^2$

③ $1+3+...+(2k-1)+(2k+1) = (k+1)^2$

$1+3+...+(2k-1) = k^2$

$1+3+...+(2k-1)+(2k+1) = k^2+2k+1$

$'' = (k+1)^2$ ✓

**61.** Use mathematical induction to prove $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+...+\dfrac{1}{2^n}=1-\left(\dfrac{1}{2}\right)^n$

① $\frac{1}{2} = 1-\left(\frac{1}{2}\right)^1$

$\frac{1}{2} = \frac{1}{2}$

② $\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^k} = 1-\left(\frac{1}{2}\right)^k$

③ $\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^k}+\frac{1}{2^{k+1}} = 1-\left(\frac{1}{2}\right)^{k+1}$

$\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^k} = 1-\left(\frac{1}{2}\right)^k$

$\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^k}+\frac{1}{2^{k+1}} = 1-\left(\frac{1}{2}\right)^k+\frac{1}{2^{k+1}}$

$'' = 1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$

$= 1-\frac{2}{2^{k+1}}+\frac{1}{2^{k+1}}$

$= 1-\frac{1}{2^{k+1}}$

$'' = 1-\left(\frac{1}{2}\right)^{k+1}$ ✓

**62.** Use mathematical induction to prove that $4^{2n}-1$ is divisible by 5, for $n \in \mathbb{Z}^+$.

① $4^2-1 = 5A$ $\quad A, B \in \mathbb{Z}$

$15 = 5A \quad A = 3$ ✓

② $4^{2k}-1 = 5A$

③ $4^{2k+2}-1 = 5B$

$4^{2k}\cdot 4^2-1 = 5B$

$(4^{2k}-1 = 5A)\cdot 4$

$4^2\cdot 4^{2k}-1\cdot 4^2 = 5A\cdot 4^2$

$4^{2k+1}-1-15 = 80A$

$4^{2k+1}-1 = 80A+15$

$4^{2k+1}-1 = 5(16A+3)$

$16A+3 = B \in \mathbb{Z}$ ✓

**63.** Use mathematical induction to show that $n! > 3$ for $n \geq 7$.

SKIP!!

64. The complex number $z$ satisfies $i(z + 2) = 1 - 2z$, where $i = \sqrt{-1}$. Write $z$ in the form $z = a + bi$, where $a$ and $b$ are real numbers.

$iz + 2i = 1 - 2z$

$2z + iz = 1 - 2i$

$z(2+i) = 1 - 2i$

$z = \frac{1-2i}{2+i}$

$z = \frac{(1-2i)(2-i)}{(2+i)(2-i)}$

$z = \frac{2 - i - 4i + 2i^2}{4 - i^2}$

$z = \frac{-5i}{5}$

$z = -i$

$\boxed{z = 0 - 1i}$

**(Total 3 marks)**

65. Given that $(a + bi)^2 = 3 + 4i$ obtain a pair of simultaneous equations involving $a$ and $b$. Hence find the two square roots of $3 + 4i$.

$a^2 + 2abi + b^2i^2 = 3 + 4i$

$a^2 - b^2 + 2abi = 3 + 4i$

$a^2 - b^2 = 3$

$2ab = 4$

$b = \frac{2}{a}$

$a^2 - \left(\frac{2}{a}\right)^2 = 3$

$\left(a^2 - \frac{4}{a^2} = 3\right) \cdot a^2$

$a^4 - 3a^2 - 4 = 0$

$(a^2 - 4)(a^2 + 1) = 0$

No sol. $a, b \in \mathbb{R}$

$a = \pm 2$

$b = \pm 1$

$\boxed{2+i, \ -2-i}$

**(Total 7 marks)**

66. Find the two square roots of $6 - 8i$. Express your answers in the form $a + bi$ where $a, b \in \mathbb{Z}$.

$a^2 - b^2 = 6$

$2ab = -8$

$b = \frac{-4}{a}$

$a^2 - \left(\frac{-4}{a}\right)^2 = 6$

$a^2 - \frac{16}{a^2} = 6$

$a^4 - 6a^2 - 16 = 0$

$(a^2 - 8)(a^2 + 2)$

$a = \pm 2\sqrt{2}$ No sol

$b = \mp \sqrt{2}$ $a, b \in \mathbb{R}$

$\boxed{\begin{array}{l} 2\sqrt{2} - \sqrt{2} \cdot i \\ -2\sqrt{2} + \sqrt{2} \cdot i \end{array}}$

**(Total 6 marks)**

67. Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.
Given that $1 + i$ and $1 - 2i$ are zeros of $p(x)$, find the values of $a$, $b$, $c$ and $d$.

CONJUGATE ROOT THM

ROOTS: $1 \pm i, \ 1 \pm 2i$

$x = 1+i \quad x = 1-i \quad x = 1+2i \quad x = 1-2i$

$(x-1-i)(x-1+i)(x-1-2i)(x-1+2i)$

$((x-1)^2 - i^2)((x-1)^2 - 4i^2)$

$(x^2 - 2x + 1 + 1)(x^2 - 2x + 1 + 4)$

$(x^2 - 2x + 2)(x^2 - 2x + 5)$ MULTIPLY

$x^4 - 2x^3 + 5x^2$

$-2x^3 + 4x^2 - 10x$

$2x^2 - 4x + 10$

$\boxed{\begin{array}{l} P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10 \\ a = -4 \quad b = 11 \quad c = -14 \quad d = 10 \end{array}}$

**(Total 7 marks)**

68. ~~The polynomial $f(x) = x^2 + 9x + 33$ has roots of $a$ and $b$. Without finding the actual roots, find the exact~~