## Comparing Equal-Tempered and Just Tuning Systems

Western music consists of 12 tones, A, A#/Bb, B, C, C#/Db, D, D#/Eb, E, F, F#/Gb, G, and G#/Ab. Each of these tones is separated by an interval called a half step. Different intervals, such as thirds and fifths, can be made using notes that are multiple half steps apart. Likewise, chords can be constructed by stacking different intervals on top of each other, using the individual frequencies to work together to create one sound. There are not definite frequencies defined for every note, although there are different systems of tuning to determine what the frequency in that specific system should be for every individual note. Two of the most commonly used tuning systems are an equal tempered system, equally spacing all notes based on one defined frequency, and a just tuning system, which defines note frequencies in a key as related by rational numbers. The purpose of this exploration is to contrast the Equal-Tempered and Just tuning systems with respect to intervals and chords, and define why the two sound different.

The reason an interval or a chord sounds the way it does is because of the constructive and destructive interference of the sound waves from the individual frequencies working together, and the resulting beat frequency. Beats are best described using an example of two tones being played simultaneously with slightly different frequencies. Consider two tones, with frequencies of 440 Hz, and 450 Hz. The graph below shows the graphs of the two individual tones laid on top of each other, with one period of the two waves starting completely in phase with each other, becoming completely out of phase in the middle, and transitioning back in phase at the end.

$$f(x) = \sin(440(2\pi)x) \text{ and } f(x) = \sin(450(2\pi)x)$$
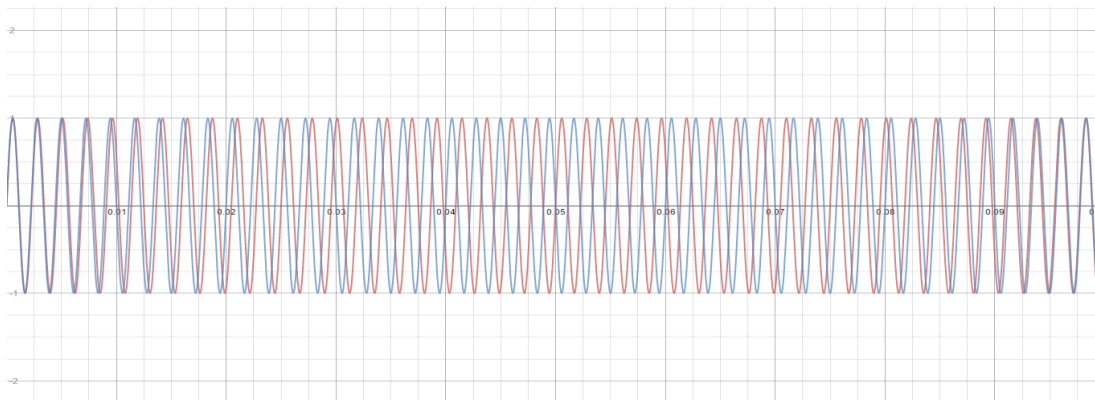
When played simultaneously, the sum of these two waves is what is heard. One period of the two waves going in and out of phase is called a beat, and is shown in the graph below.

$$f(x) = \sin(440(2\pi)x) + \sin(450(2\pi)x)$$
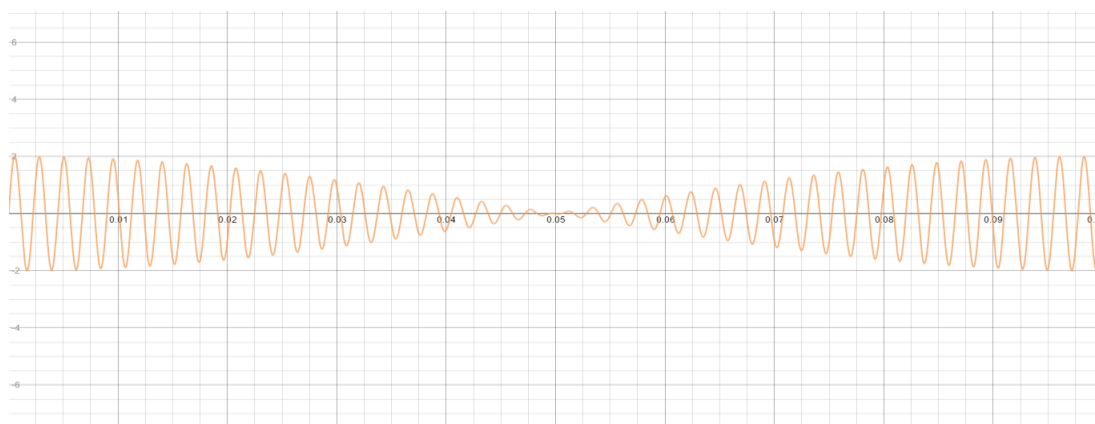
The fundamental beat frequency for two different tones being played simultaneously can be found by calculating the difference between the two frequencies. Using 440Hz and 450Hz, for example, the resulting beat frequency would be 10Hz, meaning that it would beat 10 times per second. The period of these beats can be found by taking the inverse of the beat frequency, since frequency is equal to the inverse of period.

In the Just tuning scale, all notes in the scale are related by rational numbers. Because of this, all intervals and chords using this tuning scale will have exact integer ratios. A table of these ratios is shown below.

Interval - Number of Steps Difference	Ratio to Fundamental Tone
Unison – No steps/same note	1:1
Minor Second – ½ step	25:24
Major Second – 1 step	9:8
Minor Third – 1 ½ steps	6:5
Major Third – 2 steps	5:4
Fourth – 2 ½ steps	4:3
Augmented Fourth/Diminished Fifth – 3 steps	45:32
Fifth – 3 ½ steps	3:2
Minor Sixth – 4 steps	8:5
Major Sixth – 4 ½ steps	5:3
Minor Seventh – 5 steps	9:5
Major Seventh – 5 ½ steps	15:8
Octave – 6 steps	2:1

(Scales)

This tuning system has the advantage of having exact ratios for intervals, meaning that, if correctly tuned with the system, all of the intervals will beat at significantly more natural-feeling and aurally pleasing frequencies than if tuned with a different system. As a consequence, however, an instrument that is tuned using this scale can only play in the key it is tuned to, since it is tuned with ratios to a fundamental baseline tone, which is the key the instrument is in. That is to say that if the instrument was played in a different key, the ratios of the intervals between notes in that key would not match up with the ratios the instrument is tuned to, creating unwanted dissonance, or aurally displeasing beat frequencies.

The Equal-Tempered tuning system is used as a compromise to work around the problem in the Just system of playing in different keys. This problem is alleviated by using a constant frequency multiple between notes, spacing the notes out more evenly than the spacing created by the ratios in the Just system. This allows for play in all keys to be much more equal, in that, while the overall sound for a specific key is diminished, play across different keys works just as well as in the original key, unlike the Just system. The Equal-Tempered tuning system works

based on one defined frequency, with the frequencies of all other tones being determined by a geometric series, shown below.

$$f_n = f_0 \cdot a^n$$

Where

f_n = Frequency of the note n half steps away from f_0

f_0 = Frequency of 1 fixed note that must be defined (A_4 = 440Hz is common)

$$a = 2^{\frac{1}{12}}$$

n = Number of half steps away from f_0 (Formula)

This geometric series equation to calculate frequencies is based on the fact that there are 12 tones, and going up 12 half steps results in an octave, which is the same tone with twice the frequency. The constant a accounts for this, creating consistent spacing between every tone.

Two different chord structures will now be modeled using both tuning systems in order to demonstrate the differences between the systems. An A Major chord and an A Minor chord will be modeled, using A = 440Hz as the defined tone. The Major chord is structured using the root (fundamental) tone, the major third, and the fifth, meaning that the Just system frequencies will have a 3:4:5 ratio, and the Equal-Tempered frequencies will use the defined tone, the tone 4 half steps up, and the tone 7 half steps up. The Minor chord is similarly structured, except it uses the minor third rather than the major third, giving the Just system frequencies a 10:12:15 ratio, and the Equal-Tempered frequencies the same frequencies, except the middle tone being 3 half steps up rather than 4.

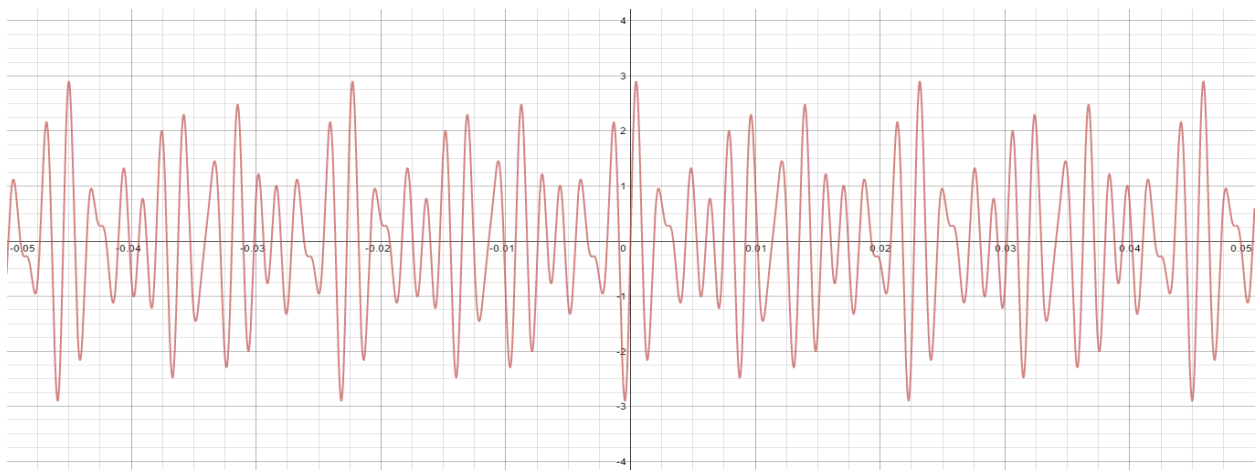**Just Tuning A Major:** $f(x) = \sin(440(2\pi)x) + \sin(550(2\pi)x) + \sin(660(2\pi)x)$
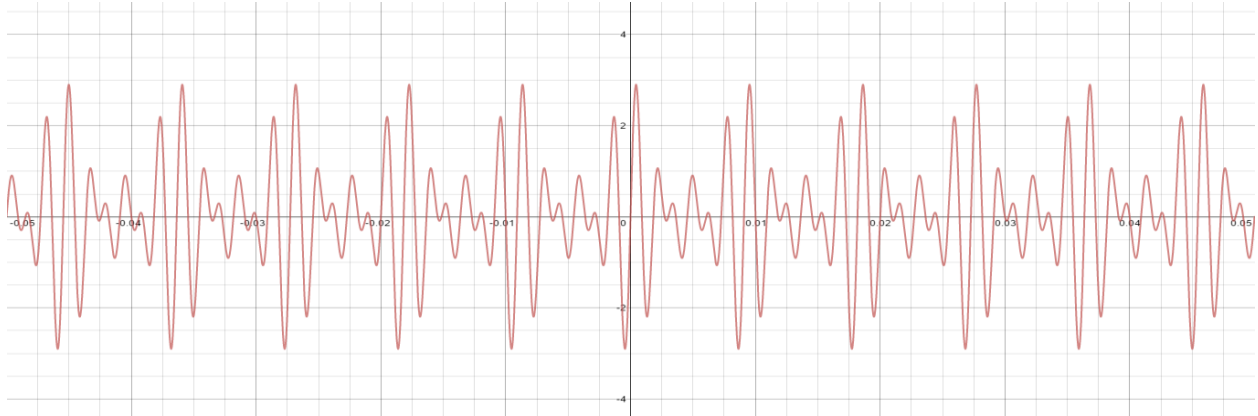
Equal-Tempered A Major:

$$f(x) = \sin(440(2\pi)x) + \sin((440 \cdot (2^{1/12})^4)(2\pi)x) + \sin((440 \cdot (2^{1/12})^7)(2\pi)x)$$
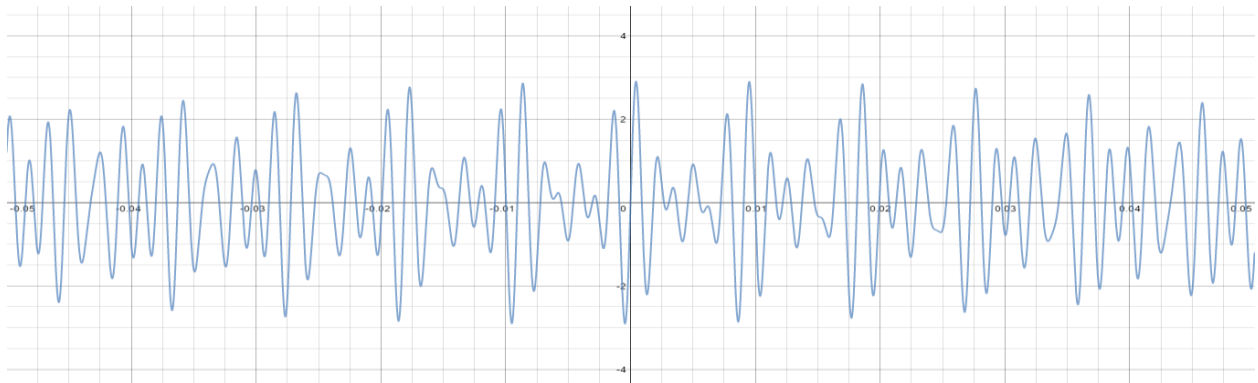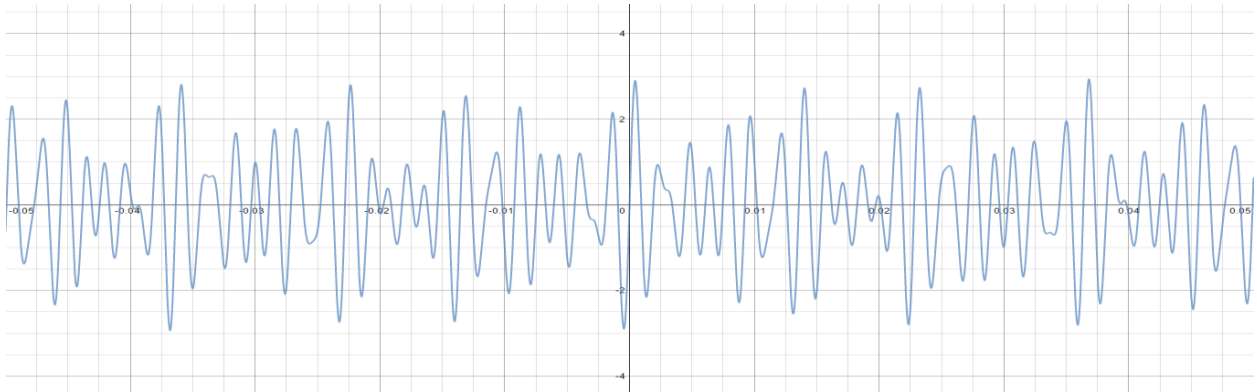

Just Tuning A Minor: $f(x) = \sin(440(2\pi)x) + \sin(528(2\pi)x) + \sin(660(2\pi)x)$

Equal-Tempered A Minor:

$$f(x) = \sin(440(2\pi)x) + \sin((440 \cdot (2^{\frac{1}{12}})^3)(2\pi)x) + \sin((440 \cdot (2^{\frac{1}{12}})^7)(2\pi)x)$$

In analyzing these graphs, a relationship was found for the Just tuned chords, that the fundamental beat frequency, or the frequency that the interference pattern repeated at, is equal to the inverse of the greatest common factor of the frequencies of the 3 tones. For the A Major chord, having frequencies of 440Hz, 550Hz, and 660Hz, the fundamental beat frequency was calculated to be $\dfrac{1}{110}$ Hz , and for the A Minor chord, having frequencies of 440Hz, 528Hz, and 660Hz, the fundamental beat frequency was calculated to be $\dfrac{1}{44}$ Hz. No such quantitative pattern was found for the equal tempered chords. Instead, although the interference pattern was still periodic, it did not repeat exactly every time. It changed slightly with every period, meaning that it never truly repeated, and it cannot be guaranteed that the function is truly periodic.

A conclusion as to how the two tuning systems sound differently with respect to chords, can be drawn based on this comparison. While the chord tuned with the Just system has a definite beat frequency and is definitely periodic, the chord tuned with the equal tempered system still does have a period, although it cannot be clearly defined, and it is not truly periodic since the interference behavior changes slightly between periods, making for a different sound

over time, a slightly less natural-feeling sound. While there is more to the picture in the behavior of the interference within the fundamental beat frequency, this conclusion does provide a baseline answer to the original question as to the difference in sounds between the two tuning systems. This conclusion can be applied by musicians and music producers, in order to create as clean of a sound as possible when creating, recording, and performing music. This has significance to me personally, as I am a musician, and plan to go into the music industry as a career, so this topic has a definite application to my interests.

Works Cited

"Formula for Frequency Table." *Formula for Frequency Table*.N.p., n.d. Web. 27 Feb. 2016.

"Scales: Just vs Equal Temperament." *Scales: Just vs Equal Temperament*. N.p., n.d. Web. 27 Feb.

   2016.

Assessment criteria:

Criterion	Comments *	Achievement level		
		Teacher	Moderator	Senior moderator
A Communication	The introduction and rationale for the paper are clearly stated. The lack of some visuals at early points make it a harder read for the non-musical. The graphs are well-designed and properly added.	0-4 3		
B Mathematical presentation	Equations and graphs are good and follow the flow of the paper. Additional visual references would have been useful, especially early on when describing keys. Equations were given using improper form.	0-3 1		
C Personal engagement	The student shows an obvious interest in the material, as shown by its quality depth and mention of potential career. Creative topic choice with good analytical thinking. Graphs were original and clear.	0-4 3		
D Reflection	The reflection was lacking in multiple ways. A connection to the graphs and what makes good sound would have been useful. Discussion of the significance of results was necessary.	0-3 1		
E Use of mathematics	Clear graphs, but where is the math? Some simple geometric sequences were added, but only a cursory mention of the periodic nature of the graphs. Where was the original math?	0-6 3		
Total:		0-20 11		