Winning a Candy Guessing Game using Volume Estimations
"Guess how many M\&M’s® are in this container to win a gift card!" Contests as such are everywhere in our society, and they offer generous rewards for simply writing a number on a piece of paper. However, in order to win these, guessing is not always the most consistent method. A little more work could go a long way. Making it a math problem gives the guesser a leg up on everyone else and could prove to be a fruitful effort.

I chose to explore this topic because too many people take the wrong approach when participating in these contests. They look at the container for a couple seconds, write down a wild guess, and walk away. This makes no sense to me. Many contests offer very generous prizes for whoever has the closest guess to the actual number, so why not spend a little time and try to calculate the correct answer with math? I believe that anyone who utilizes math in these contests gives themself a substantial advantage over the other participants, and I aim to prove that in this paper.

For this exploration, I will attempt to correctly guess the number of $\mathrm{M} \& \mathrm{M}{ }^{\wedge}{ }^{\circledR}$ ® in a completely filledcylindrical vase using volumes by rotation to estimate the volume of the $M \& M$ 's®. I will then compare the volume found by rotation to the volume found using the formula for the volume of an ellipsoid. I will also compare the calculated number of M\&M's® to the actual number found when individually counting them. This will allow me to form a conclusion on the effectiveness of this technique in guessing contests.

The basic premise of this problem is a simple one. To determine the number of $\mathrm{M} \& \mathrm{M}{ }^{\prime}{ }^{\circledR}{ }^{\circledR}$ in the container, I must find the volume of the container and the volume of an individual $M \& M ®$, and divide the container volume by the individual volume. To begin solving the problem, I found the volume of the glass vase. I measured the diameter and height of the vase in
millimeters using a metric ruler. I substituted in my measured values for radius ( 56.5 mm ) and height ( 332 mm ) into the volume equation and calculated the volume:

$$
\begin{gathered}
V=r^{2} h \\
V=(56.5)^{2}(332) \\
V=3329544.717276 \mathrm{~mm}^{3} .
\end{gathered}
$$

One important factor to consider with the volume of the container is packing density. When filling a container with $M \& M ' s ®$, the $M \& M ' s ®$ will not take up all of the volume of the container. There will be pockets of air between the $M \& M$ 's® since they do not pack perfectly together. To compensate for this, I will utilize a packing fraction. This is the approximate percentage of the total volume that will be occupied by the M\&M's®. Because each arrangement of $\mathrm{M} \& \mathrm{M}{ }^{\prime}{ }^{\circledR}$ ® in a container is random, the packing fraction varies. A study at Cornell University ran a simulation to determine packing density for spheroids that are similar in size to M\&M's ${ }^{\circledR}$ and found that the fraction generally varies from $=0.68 \quad 0.71$ (Donev).The arrangement I examined was packed randomly, so I used the average value for the packing fraction: $=0.695$. I then calculated the volume of the container occupied by M\&M's® using the equation

$$
V_{F}=V_{T}
$$

where $V_{F}$ is the volume filled by $\mathrm{M} \& \mathrm{M}{ }^{\prime}{ }^{\circledR}$ ® and $V_{T}$ is the total volume of the container. I substituted in for $V_{T}$ and

$$
\begin{gathered}
V_{F}=(3329544.717276)(0.695) \\
V_{F}=2314033.578509 \mathrm{~mm}^{3}
\end{gathered}
$$

and found that the volume occupied by M\&M's® in the cylindrical container is $2314033.578509 \mathrm{~mm}^{3}$. It is important to note that I have carried six digits after the decimal to ensure that rounding will not significantly change my calculated values.

Next, I determined the volume of a single $M \& M ®$ using a volume by rotation calculation. To do this, I modeled the curve of an $M \& M ®$ in the first quadrant of a graph. Next, I rotated it about the y-axis to maintain an ellipsoid shape. Finally, I doubled the volume from the rotation to account for the portion of the volume that is below the x -axis. This process proved to be tricky, however, because small discrepancies in measurements became largely amplified due to the sheer number of $M \& M$ 's® in the container. Initially, I attempted to estimate the dimensions of an $M \& M ®$ and model it graphically using the square root equation

$$
y=\frac{2.5}{\sqrt{6}} \sqrt{6 x}
$$

where the total height of the $\mathrm{M} \& \mathrm{M} ®$ was 5 mm and the total length was 12 mm . I then solved for $x$ in terms of $y$

$$
x=6 \quad \frac{6 y^{2}}{6.25} .
$$

Using this equation, I performed a rotation about the $y$-axis. The general formula of a rotation is given by

$$
V={ }_{a}^{b}(f(y))^{2} d y
$$

where $f(y)$ is the radius of the "disk" that is formed by rotation and $a$ and $b$ are the lower and upper boundaries of the rotation. Plugging in my equation and boundaries, I found the volume

$$
V=\int_{0}^{2.5}\left(6 \frac{6 y^{2}}{6.25}\right)^{2} d y
$$

$$
\begin{gathered}
V=\left[\begin{array}{ll}
6 y & \frac{2 y^{3}}{6.25}
\end{array}\right]_{0}^{2.5} \\
V=10 \mathrm{~mm}^{3}
\end{gathered}
$$

Finally, I multiplied that volume by two and found the volume of a single $\mathrm{M} \& \mathrm{M} ®$ to be $20 \mathrm{~mm}^{3}$. I then calculated the number of M\&M's® in the jar ( $m$ )

$$
\begin{gathered}
m=\frac{V_{F}}{V_{M \& M}} \\
m=\frac{2314033.578509}{20} \\
m=36,829 \mathrm{M} \& \mathrm{M}^{\prime} \mathrm{S} ® .
\end{gathered}
$$

This result is incredibly too high for a container of the size that I used. To fix this, I decided to model the shape of an $M \& M ®$ using an equation of an ellipse instead. Maintaining the same dimensions, I formulated the equation

$$
\frac{x^{2}}{36}+\frac{y^{2}}{6.25}=1
$$

The following graph shows the difference between the two equations and demonstratesthat the second equation better models the shape of an M\&M® ("Desmos"):


Comparison of Different Shapes to Model an M\&M®
It is also easy to see in the graph why the square root equation would yield a smaller volume than the ellipse equation, leading to a much larger number of $M \& M^{\prime}{ }^{\circledR}$ ®. Once I decided to use the ellipse equation, I solved for x in terms of y

$$
x=\sqrt{\frac{22536 y^{2}}{6.25}}
$$

In the above equation, I left off the plus/minus symbol in front of the square root symbol because I am only focusing on the portion of the graph in the first quadrant. I performed the rotation of that equation about the $y$-axis

$$
\begin{gathered}
V=\int_{0}^{2.5}\left(\sqrt{\frac{225 \quad 36 y^{2}}{6.25}}\right)^{2} d y \\
V=\left[\begin{array}{ll}
36 y & 1.92 y^{3}
\end{array}\right]_{0}^{2.5} \\
V=60 \mathrm{~mm}^{3} .
\end{gathered}
$$

I multiplied that volume by two and found the volume of a single $M \& M ®$ to be $V=120 \mathrm{~mm}^{3}$. I used this value to calculate the number of $M \& M^{\prime}{ }^{\wedge}{ }^{\circledR}$ in the jar

$$
\begin{aligned}
m & =\frac{2314033.578509}{120} \\
m & =6,138 \mathrm{M} \& \mathrm{M}^{\prime} \mathrm{s} ® .
\end{aligned}
$$

This result is significantly better than the result found using a square root equation. However, it is still much too high for the container. Also, It is important to note that I have rounded each value of the number of $M \& M^{\prime}{ }^{\circledR}{ }^{\circledR}$ to the nearest whole number. I did this because most contests do not count smaller or broken M\&M's® as partial. Each piece counts as a whole.

In order to get a more realistic number, I needed to find more accurate dimensions of $\mathrm{M} \& \mathrm{M}{ }^{\prime}{ }^{\circledR}$. To do that, I used a dial caliper and measured the length and height of ten M\&M's®. I took the average for both measurements for the ten trials because $\mathrm{M} \& \mathrm{M}$ 's® vary in size.

| M\&M® | Length (mm) | Height (mm) |
| :--- | :--- | :--- |
| 1 | 13.0 | 6.7 |
| 2 | 13.2 | 7.0 |
| 3 | 13.1 | 6.5 |
| 4 | 13.1 | 7.0 |
| 5 | 13.1 | 6.6 |
| 6 | 13.0 | 6.9 |
| 7 | 13.5 | 6.5 |
| 8 | 13.1 | 7.0 |
| 9 | 13.4 | 6.8 |
| 10 | 13.0 |  |


| AVERAGE | 13.2 | 6.8 |
| :--- | :--- | :--- |

It is also important to note here that I rounded the averages to the tenths place because that is the number of significant figures given on the caliper. I used these new dimensions in the ellipse equation and modeled the $\mathrm{M} \& \mathrm{M} ®$ by the equation

$$
\frac{x^{2}}{43.56}+\frac{y^{2}}{11.56}=1
$$

Solving for $x$ in terms of $y$, the equation of the $M \& M ®$ in the first quadrant is given by

$$
x=\sqrt{\frac{503.553643 .56 y^{2}}{11.56}} .
$$

I repeated the same process of rotating the equation about the $y$-axis to find the volume

$$
\begin{gathered}
V=\int_{0}^{3.4}\left(\sqrt{\frac{503.553643 .56 y^{2}}{11.56}}\right)^{2} d y \\
V=\left[43.56 y 1.256055 y^{3}\right]_{0}^{3.4} \\
V=98.736 \mathrm{~mm}^{3}
\end{gathered}
$$

Multiplying by two, I found the volume of a single $M \& M ®$ to be $V=197.472 \mathrm{~mm}^{3}$. Finally, I calculated the number of M\&M's® in the container

$$
\begin{aligned}
& m=\frac{2314033.578509}{197.472} \\
& m=3,730 \mathrm{M} \& \mathrm{M}^{\prime} \mathrm{s} ® .
\end{aligned}
$$

My final guess for the number of $\mathrm{M} \& \mathrm{M}$ 's ${ }^{\circledR}$ in the vase was $3,730 \mathrm{M} \& \mathrm{M}{ }^{\prime} \mathrm{s}^{\circledR}$.
For a comparison, I also calculated the volume of a single $\mathrm{M} \& \mathrm{M}$ ®using the formula for the volume of an ellipsoid

$$
V=\frac{4}{3} a b c
$$

where $a, b$, and $c$ are the three axes of the ellipsoid. Plugging in my measurements, I found the volume using this formula

$$
\begin{gathered}
V=\frac{4}{3}(6.6)(6.6)(3.4) \\
V=197.472 \mathrm{~mm}^{3} .
\end{gathered}
$$

This formula yields the same volume for an $M \& M ®$ as the volume by rotation, demonstrating that performing a rotation to determine the volume of an $\mathrm{M} \mathrm{\& M} \mathrm{~B}^{8}$ is just as effective and precise as using the volume formula.

As it turns out, my final guess of $3,730 \mathrm{M} \& \mathrm{M}$ ' ${ }^{\circledR}$ ®was very close to the actual number. After painstakingly counting every single $\mathrm{M} \& \mathrm{M} ®, \mathrm{I}$ found that there were $3,704 \mathrm{M} \& \mathrm{M}{ }^{\prime}{ }^{\circledR}{ }^{\circledR}$ in the container. Calculating percent error, I found that my answer was within one percent of the actual value:

$$
\begin{gathered}
\%_{\text {error }}=\frac{37303704}{3704} \\
\%_{\text {error }}=0.702 \%
\end{gathered}
$$

Therefore, I can conclude that this method is a successful method for estimating the number of M\&M's® in a container. Being this close to the actual answer will almost always win the contest. I showed my container to multiple people and asked them to guess the number. Most people said a number between 500 and 1,000 , while the closest guess of all was 3,000 . If I were to enter my number into a contest with the people I asked, I would have won easily. However, this method does still have its weaknesses.

My method of estimating the number of $M \& M ' s{ }^{\circledR}$ in a container using volumes of rotation yielded an accurate result, but it is not without its limitations. Mainly, the method relies on knowing precise measurements of both the container and the object in it. If the contest uses M\&M's® or another common object, the guesser can get that type of objectand measure it beforehand. The harder part is measuring the container. Most contests will not allow the guesser to come in with a ruler and take measurements on the container, as it will give that person an unfair advantage over others. However, the guesser could reasonably estimate the dimensions of the container and still have a distinct advantage over all the other participants. Unlike with the volume of the objects, the volume of the container is not as sensitive to slight fluctuations in the dimensions, so estimating its dimensions should yield fairly accurate results.

Another limitation of this investigation is that it is only effective if the container and the objects are of uniform shape. Alternative methods such as completely filling the container with water and measuring the volume of water could be used if the container is not a simple shape, but this is not possible for a contest. Similarly, my method would not work if the objects in the container were different shapes and sizes. At that point, the guesser would truly be a guesser, not an estimator.

Finally, my method relies on knowing the packing factor of the objects. These are easy to look up for common shapes such as ellipsoids or spheres, but finding the value for uncommon objects can be impossible. In this case, the guesser would have to choose an arbitrary valuefor the objects. Based on my research, a value between 0.6 and 0.7 is the best bet when choosing an arbitrary value. However, there is a big difference between the two calculated volumes using those factors, so there is a large degree of luck involved when choosing an arbitrary value.

Despite these limitations, my method still gives the guesser a much greater chance at winning the prize than by simply guessing. If the guesser can obtain measurements of the container and the object, he can come up with a nearly exact answer with a few calculations.

Even if he does not have precise measurements, he can still make a reasonable guess that, with a little luck, could win the contest and whatever reward that is offered. This goes to show that a little math in these contests can go a long way.

## Works Cited

"Desmos Graphing Calculator." Desmos Graphing Calculator. N.p., n.d. Web. 26 Feb. 2016. Donev, Aleksandar. "Improving the Density of Jammed Disordered Packings Using Ellipsoids." Improving the Density of Jammed Disordered Packings Using Ellipsoids (n.d.): n. pag. Cornell University. Web. 26 Feb. 2016.

Appendix


Side View of Glass Vase filled with M\&M's ${ }^{\circledR}$


Top View of Glass Vase filled with $M \& M^{\prime}{ }^{\circledR}{ }^{\circledR}$

## Assessment criteria:

## Criterion

## Achievement level

Teacher Moderator | Senior |
| :---: |
| moderator |

The goal of this exploration was clear throughout, and the paper was
A Communication reasons for changes along the way were clearly articulated.

0-4
4 $\square$
$\square$

Equations were formatted well, and a simple graph was added to show difference in areas of the equations used. The images at the end would have been better served in the paper and not as appendices.

The paper was very original in topic selection and the implementation of the packing fraction idea. The student made conjectures, tested them, and adjusted along the way when things seemed off.

D Reflection
Limitations were discussed on multiple occasions, and the student
showed a genuine interest in the topic. Possible extensions or
applications would have been useful, possibly to other fields of study.

The use of mathematics was correct, relevant to the topic, and varied from simple volume to Calculus. The usage showed an excellent level of sophistication and understanding, while staying on topic.

## 0-3

2 $\square$
$\square$
0.3

2

$\square$
$0-4$
4


0-6
6 $\square$
$\square$

0-20
Total: $\square$ 18 $\square$
$\square$

