Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 7]

The function $f$ is defined by $f(x) = \mathrm{e}^{-x}\cos x + x - 1$.

By finding a suitable number of derivatives of $f$, determine the first non-zero term in its Maclaurin series.

**2.** [Maximum mark: 8]

(a) Show that $y = \frac{1}{x}\int f(x)\,\mathrm{d}x$ is a solution of the differential equation

$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x)$, $x > 0$. [3]

(b) Hence solve $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^{-\frac{1}{2}}$, $x > 0$, given that $y = 2$ when $x = 4$. [5]

**3.** [Maximum mark: 17]

(a) Show that the series $\sum_{n=2}^{\infty}\frac{1}{n^2\ln n}$ converges. [3]

(b) (i) Show that $\ln(n) + \ln\left(1 + \frac{1}{n}\right) = \ln(n+1)$.

(ii) Using this result, show that an application of the ratio test fails to determine whether or not $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ converges. [6]

(c) (i) State why the integral test can be used to determine the convergence or divergence of $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$.

(ii) Hence determine the convergence or divergence of $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$. [8]

**4.** [Maximum mark: 12]

(a) Use l'Hôpital's rule to find $\lim_{x\to\infty} x^2 e^{-x}$. [4]

(b) Show that the improper integral $\int_0^\infty x^2 e^{-x}\,dx$ converges, and state its value. [8]

**5.** [Maximum mark: 16]

(a) The mean value theorem states that if $f$ is a continuous function on $[a, b]$ and differentiable on $]a, b[$ then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some $c \in ]a, b[$.

(i) Find the two possible values of $c$ for the function defined by $f(x) = x^3 + 3x^2 - 2$ on the interval $[-3, 1]$.

(ii) Illustrate this result graphically. [7]

(b) (i) The function $f$ is continuous on $[a, b]$, differentiable on $]a, b[$ and $f'(x) = 0$ for all $x \in ]a, b[$. Show that $f(x)$ is constant on $[a, b]$.

(ii) Hence, prove that for $x \in [0, 1]$, $2\arccos x + \arccos\left(1 - 2x^2\right) = \pi$. [9]