Summary of Convergence Tests for Infinite Series

- Infinite Geometric Series .... $u_k = (r)^k, |r| < 1 \longrightarrow$ Converges
- Divergence Test .... $\lim_{k\to\infty} u_k \neq 0 \longrightarrow$ Diverges
- P-Series Test .... $\sum_{x=1}^{\infty} \frac{1}{x^p}, p > 1 \longrightarrow$ Converges
- Comparison Test .... $a_k \le b_k$ → If b_k Converges, a_k Converges; → If a_k Diverges, b_k Diverges
- Limit Comparison Test .... $\lim_{k\to\infty} \frac{a_k}{b_k} = l > 0 \longrightarrow a_k, b_k$ Converge/Diverge together
- Integral Test .... If $\int_1^{\infty} f(x)\cdot dx$ Converges, then $\sum_{k=1}^{\infty} f(k)$ Converges
- Alternating Series Test .... Converges if: $\lim_{k\to\infty}|u_k| = 0$; $|u_{k+1}| < |u_k|$
- Ratio Test ....
  - $\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| < 1 \longrightarrow$ Absolutely Convergent
  - $\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| > 1 \longrightarrow$ Divergent
  - $\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = 1 \longrightarrow$ Ratio Test Inconclusive

Tips/Rules for Convergence Tests

1. Divergence Test $\longrightarrow \lim_{x\to\infty} u_k \neq 0$
2. Check for: p-series or geometric
   $\sum_{x=1}^{\infty} \frac{1}{x^p}$ $(p > 1)$ $\qquad \sum_{x=0}^{\infty} a(r)^x$ or $\sum_{x=1}^{\infty} a(r)^{x-1}$ $(|r| < 1)$
3. Polynomials in fraction $\longrightarrow$ Comparison/Limit Comparison (Terms need to be all positive)
4. Factorials or constants to a power $\longrightarrow$ Ratio Test
5. $(-1)^x$ or $(-1)^{x-1}$ or $\cos(\pi x)$ $\longrightarrow$ Alternating Series Test
6. Possible u-substitution or easy form $\longrightarrow$ Integral Test

1.) $\sum_{k=2}^{\infty} \frac{\sqrt{k^3-6}}{k^3+2}$ → ACTS LIKE $\frac{\sqrt{k^3}}{k^3} = \frac{1}{k^{3/2}}$

$\frac{\sqrt{k^3-6}}{k^3+2} \le \frac{\sqrt{k^3}}{k^3} \le \frac{1}{k^{3/2}}$

NUM. BIGGER, DENOM. SMALLER

CONVERGES $\left(\frac{1}{x^p}, p>1\right)$

CONVERGES BY COMPARISON TEST.

2.) $\sum_{x=1}^{\infty} \frac{k^6}{(1-2k^3)^2}$

$\frac{k^6}{1-4k^3+4k^6}$ → ACTS LIKE $\frac{k^6}{k^6}$

$\lim_{k\to\infty} = \frac{1}{4} \neq 0$

DIVERGES BY DIVERGENCE TEST.

3.) $\sum_{k=3}^{\infty} \frac{\sqrt{k}+k}{2\sqrt{k}+k^3-4}$ → ACTS LIKE $\frac{1}{k^2}$

COMPARISON TEST WILL NOT WORK ON ITS OWN

$\frac{\sqrt{k}-k}{2\sqrt{k}+k^3-4} \le \frac{\sqrt{k}}{2\sqrt{k}-4}$ STUCK...

LIMIT COMPARISON TEST

$\lim_{k\to\infty} \frac{\frac{1}{k^2}}{\frac{\sqrt{k}+k}{2\sqrt{k}+k^3-4}} = \frac{k^3+2\sqrt{k}-4}{k^3+k^{5/2}} = \boxed{1>0}$ ✓

SO, THEY ACT TOGETHER. $\frac{1}{k^2}$ CONVERGES $\left(\frac{1}{x^p}, p>1\right)$.

$\therefore \frac{\sqrt{k}-k}{2\sqrt{k}+k^3-4}$ CONVERGES BY LIMIT COMPARISON TEST.

4.) $\sum_{k=2}^{\infty} \frac{k^2}{(2k-1)!}$ RATIO TEST

$u_k = \frac{k^2}{(2k-1)!} \qquad u_{k+1} = \frac{(k+1)^2}{(2(k+1)-1)!} = \frac{(k+1)^2}{(2k+1)!}$

$\lim_{k\to\infty} \frac{(k+1)^2\cdot(2k-1)!}{k^2\cdot(2k+1)!} = \left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{(2k+1)(2k)}$ → $1 \cdot 0$

$= 0 < 1$ ✓ ABSOLUTE CONVERGENCE

CONVERGES BY RATIO TEST.

3C - Power Series

A power series is an infinite series in the form:

$$f(x)=\sum_{k=0}^{\infty}a_k(x-b)^k=a_0+a_1(x-b)+a_2(x-b)^2+...$$

A geometric series is an example of a power series where $b=0$ and a_k is a constant, a, such that:

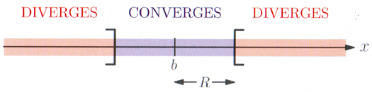
$$\sum_{k=0}^{\infty}a\cdot x^k=a+ax+ax^2+ax^3+...=\frac{a}{1-x}$$

Converges if $|x|<1$

$-1<x<1$

Interval of Convergence

$-1<x<1$



$R=1$

Radius of Convergence

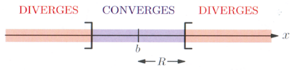
$$\sum_{k=0}^{\infty}a\cdot x^k=a+ax+ax^2+ax^3+...=\frac{a}{1-x}$$

Interval of Convergence

$-1<x<1$

Radius of Convergence

$R=1$



If you plug in any value for x that is on the interval of convergence, you will get a finite sum for the series.

Find the interval and radius of convergence for the series below.

$$\sum_{k=0}^{\infty}4(x-5)^k=4+4(x-5)+4(x-5)^2+...$$

Interval of Convergence

Radius of Convergence

Rewrite the function below as an infinite series, listing the interval of convergence and radius of convergence.

$$g(x)=2-8x^2+32x^4-128x^6+...$$

Interval of Convergence

Radius of Convergence

Rewrite the function below as an infinite series, listing the interval of convergence and radius of convergence.

$$f(x)=\frac{1}{3+x^2} \longrightarrow S=\frac{a_1}{1-r}$$

$$\frac{1}{3+x^2}=\sum_{k=0}^{\infty}\frac{1}{3}\left(-\frac{x^2}{3}\right)^k=\frac{1}{3}x^0-\frac{1}{9}x^2+\frac{1}{27}x^4-\frac{1}{81}x^6+...$$

Interval of Convergence

Radius of Convergence

An alternate way to find the interval of convergence is to apply the ratio test.

KEY POINT 3.11

Ratio Test

Given a series $\sum_{k=1}^{\infty}u_k$, if:

- $\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|<1$, then the series is absolutely convergent (and hence convergent)
- $\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|>1$, then the series is divergent
- $\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|=1$, then the Ratio Test is inconclusive.

For what values of x is the series below convergent?

(Hint: Be careful to check to check the boundaries of your interval of convergence!)

$$\sum_{k=0}^{\infty} \frac{(x-2)^k}{\sqrt{k+1}}$$

$$|x-2|<1$$

$$-1<x-2<1$$

$$\boxed{1<x<3}$$ → What happens at $x = 1$ and $x = 3$?

3C p.92 #1-7

-3C Short Quiz on Wednesday 4/6

Key Point 3.15 (p.91)

A power series can be differentiated or integrated term by term over any interval contained within its interval of convergence (and possibly at the endpoints of the interval).

This characteristic will be utilized next Chapter when we look at different types of power series.

1. Find the radius of convergence and give the interval of convergence of the following:

(a) (i) $\sum_{k=0}^{\infty} k!x^k$ (ii) $\sum_{k=0}^{\infty} k!(x+3)^k$

1. Find the radius of convergence and give the interval of convergence of the following:

(b) (i) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ (ii) $\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{k!}$

1. Find the radius of convergence and give the interval of convergence of the following:

(c) (i) $\sum_{k=0}^{\infty} \frac{(-1)^k k x^k}{(k+1)5^k}$ (ii) $\sum_{k=0}^{\infty} \frac{x^{2k}}{\sqrt{k+1}}$

1. Find the radius of convergence and give the interval of convergence of the following:

(d) (i) $\sum_{k=1}^{\infty} \frac{(x-5)^k}{\sqrt{k}\,2^k}$ (ii) $\sum_{k=0}^{\infty} \frac{(x-1)^k}{k^2}$

2. Find the values of x for which the following converge:

(a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$

(b) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

(c) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

3. Find the values of x for which the power series converges.

$$\sum_{k=0}^{\infty} \frac{k^4 x^{k+1}}{(k+1)!}$$

4. Find the interval of convergence for the power series:

$$\sum_{k=1}^{\infty} 5k(x+4)^k$$

5. (a) Find the interval of convergence for the power series

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

(b) Find $f'(x)$ stating for which values of x this holds.

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$f'(x) = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$f''(x) = 0 + 0 + 1 + x + \frac{x^2}{2!} + \ldots$$

6. For the series $\sum_{k=0}^{\infty} \frac{(x+3)^k}{(k+1)4^k}$

(a) Find the radius of convergence.

(b) Find the values of x for which the series converges.

7. Find the interval of convergence of the following, discussing the nature of any convergence at the end points of the interval.

(a) $\sum_{k=0}^{\infty} \frac{(-1)^k kx^k}{(k+1)4^k}$

(b) $\sum_{k=0}^{\infty} \frac{(-2)^k x^k}{k^5}$

(c) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{5k+1}}{5k+1}$