3B - Convergent and Divergent Series

$$\int_1^\infty \frac{1}{x}\cdot dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\cdot dx$$
$$= \lim_{b\to\infty}\left[\ln x\right]_1^b$$
$$= \lim_{b\to\infty}(\ln b - \ln 1)$$
$$= \lim_{b\to\infty}(\ln b)$$
$$= \infty \quad \text{Divergent}$$

$$\int_1^\infty \frac{1}{x^2}\cdot dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\cdot dx$$
$$= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$$
$$= \lim_{b\to\infty}(-\frac{1}{b}+1)$$
$$= 1 \quad \text{Convergent}$$

$\sum_{k=1}^{\infty}\frac{1}{k}$ Harmonic Series

$\lim_{b\to\infty}\int_1^b \frac{1}{x}dx = \infty$ Divergent

$\sum_{k=1}^{\infty}\frac{1}{k^2}$

$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = 1$ Convergent

Absolute vs. Conditional Convergence

$\sum_{k=1}^{\infty} u_k$ Alternating series — Easier rules for convergence

$\sum_{k=1}^{\infty} |u_k|$ Alternating series values w/o alternating signs — Stricter rules for convergence

If $\sum_{k=1}^{\infty}|u_k|$ is convergent, then the series $\sum_{k=1}^{\infty}u_k$ is <u>*absolutely convergent*</u>.

If $\sum_{k=1}^{\infty}|u_k|$ is divergent but $\sum_{k=1}^{\infty}u_k$ is convergent, then the series $\sum_{k=1}^{\infty}u_k$ is <u>*condtionally convergent*</u>.

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$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ $\quad |u_k| = \frac{1}{k}$ $\quad |u_{k+1}| = \frac{1}{k+1}$

$\sum_{k=1}^{\infty}\frac{1}{k} \longrightarrow \sum_{k=1}^{\infty}\frac{1}{k^p}, p \not> 1 \longrightarrow$ Divergent

$\sum_{k=1}^{\infty}\frac{(-1)^k}{k} \longrightarrow \lim_{k\to\infty}|u_k| = 0,\ |u_{k+1}| < |u_k| \longrightarrow$ Convergent

$\therefore \sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ is a <u>***conditionally convergent***</u> alternating series.

Examples

$u_k = \sum_{k=1}^{\infty}(-1)^k\sqrt{k}$ Divergent	$\lvert u_k\rvert = \sum_{k=1}^{\infty}\sqrt{k}$ Divergent	Divergent
$u_k = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ Convergent	$\lvert u_k\rvert = \sum_{k=1}^{\infty}\frac{1}{k}$ Divergent	Conditionally Convergent
$u_k = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}$ Convergent	$\lvert u_k\rvert = \sum_{k=1}^{\infty}\frac{1}{k^2}$ Convergent	Absolutely Convergent

Absolute Convergence - Extension

If a series is absolutely convergent, then it is convergent.

Thus, if $\sum_{k=1}^{\infty}|u_k|$ is convergent then so is $\sum_{k=1}^{\infty}u_k$.

6. Determine whether the following converge absolutely, conditionally or diverge.

(a) (i) $\sum_{k=1}^{\infty}\frac{\cos k}{k^2}$ (ii) $\sum_{k=1}^{\infty}\frac{\sin k}{k^3+1}$

(b) (i) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{2k+1}$ (ii) $\sum_{k=1}^{\infty}(-1)^k\frac{4k^2+3}{3k-1}$

(c) (i) $\sum_{k=2}^{\infty}\frac{(-1)^k k}{k^2+1}$ (ii) $\sum_{k=2}^{\infty}\frac{(-1)^k}{k^2-1}$

(d) (i) $\sum_{k=1}^{\infty}\frac{\cos(k\pi)}{\sqrt{k+3k-1}}$ (ii) $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt[4]{k^2+2k+1}}$

(e) (i) $\sum_{k=1}^{\infty}\frac{(-2)^{k+1}}{(2k+1)!}$ (ii) $\sum_{k=2}^{\infty}(-1)^k\frac{k^4}{e^k}$