Integral Test

Given a positive decreasing function $f(x), x \geq 1$,

- If $\int_1^{\infty} f(x)dx$ is convergent then $\sum_{k=1}^{\infty} f(k)$ is convergent.
- If $\int_1^{\infty} f(x)dx$ is divergent then $\sum_{k=1}^{\infty} f(k)$ is divergent.

Integral Test

$$\int_1^{\infty} \frac{1}{x^p} dx$$

The integral is convergent if:

$$\frac{1}{x^p}, p > 1$$

$\int_1^{\infty} \frac{1}{x} \cdot dx \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k}$ The integral is divergent, therefore the series is divergent.

$\int_1^{\infty} \frac{1}{x^2} \cdot dx \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2}$ The integral is convergent, therefore the series is convergent.

4. Using the Integral Test, determine whether the following converge or diverge.

(a) (i) $\sum_{k=1}^{\infty} \frac{1}{k+2}$ (ii) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k+3}}$

(b) (i) $\sum_{k=4}^{\infty} \frac{k}{(k^2+9)^{\frac{3}{2}}}$ (ii) $\sum_{k=4}^{\infty} \frac{k}{k^2+9}$

(c) (i) $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ (ii) $\sum_{k=1}^{\infty} \frac{3}{2k^2+5}$

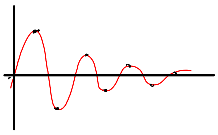
(d) (i) $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{1}{k}\right)$ (ii) $\sum_{k=1}^{\infty} \frac{1}{k^3} \sin\left(\frac{1}{k^2}\right)$

(e) (i) $\sum_{k=2}^{\infty} \frac{\ln k}{k}$ (ii) $\sum_{k=1}^{\infty} k e^{-k^2}$

Alternating Series Test

Alternating Series = Alternates back and forth between positive and negative values

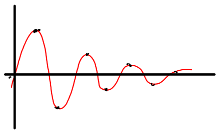
For an alternating series $\sum_{k=1}^{\infty} u_k$, the series is convergent if:



- $\lim_{k \to \infty} |u_k| = 0$
- $|u_{k+1}| \leq |u_k|$ for sufficiently large k

Alternating Series Test

Values must get progressively smaller as x increases and they must approach 0. The criteria for an alternating series to converge is not as strict as those for non-alternating series. See the example below.



$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

1) $|u_k| = \frac{1}{k}, |u_{k+1}| = \frac{1}{k+1}$ --------> $k+1 > k$, $\frac{1}{k+1} < \frac{1}{k}$

2) $\lim_{k \to \infty} \frac{1}{k} = 0$

Therefore, by the alternating series test,

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is convergent.

Note: This series is convergent even though the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is not.

Truncation Error for Alternating Series

If $S = \sum_{k=1}^{\infty} u_k$ is the sum of an alternating series that satisfies the conditions:

- $\lim_{k \to \infty} |u_k| = 0$
- $|u_{k+1}| \le |u_k|$ for all $k \in \mathbb{Z}^+$

then the error in taking the first n terms (S_k) as an approximation to S is less than the absolute value of the $(k+1)$th term.

$$|S - S_k| < |u_{k+1}|$$

Example of Truncation Error for Alternating Series

How many terms of the series $\sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{1}{k^2}$ is it necessary to take to find an approximation accurate to within 0.001?

5. Use the Alternating Series Test to determine whether the following series converge or not. Where they do, find an upper bound (to 3SF) on the error in taking the first 10 terms of the series as an approximation.

(a) (i) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{\frac{2}{3}}}$ (ii) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

(b) (i) $\sum_{k=1}^{\infty} (-1)^k \frac{2k+3}{3k+4}$ (ii) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k-1}{k}$

(c) (i) $\sum_{k=2}^{\infty} \frac{(-2)^k}{3^k}$ (ii) $\sum_{k=2}^{\infty} \frac{(-3)^k}{2^k}$

(d) (i) $\sum_{k=2}^{\infty} \frac{\cos(k\pi)}{\ln k}$ (ii) $\sum_{k=1}^{\infty} \frac{\cos(k\pi)k}{k!}$

(e) (i) $\sum_{k=1}^{\infty} \frac{(-k)^{k+1}}{k^2}$ (ii) $\sum_{k=1}^{\infty} \frac{(-k)^{k+1}}{k^{3k}}$