3A - Important Topics

★ If the sequence of partial sums $S_1, S_2, S_3, \ldots$ converges to a limit S, then the infinite series is convergent (to S).

$$S = \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} u_k$$

Essential Question:

How do we tell if an infinite series converges?

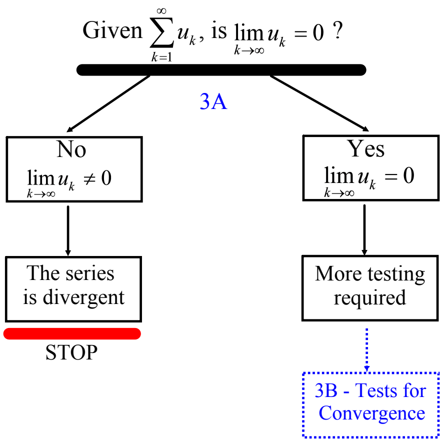
Convergent and Divergent Series

★ If a series $\sum_{k=1}^{\infty} u_k$ is convergent then $\lim_{k\to\infty} u_k = 0$.

★ However, if $\lim_{k\to\infty} u_k = 0$ then $\sum_{k=1}^{\infty} u_k$ is not necessarily convergent.

★ If $\lim_{k\to\infty} u_k \neq 0$ or the limit does not exist, then $\sum_{k=1}^{\infty} u_k$ divergent.

Convergence/Divergence Flow Chart



Ex #1

Show that the series $\sum_{k=1}^{\infty} \frac{k^2+3k+1}{4k^2+3}$ diverges.

Ex #2:

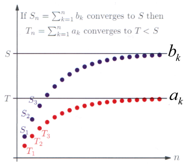
Determine if the series below converges or diverges.

$$\sum_{k=1}^{\infty} \frac{4k^2-k+5}{k^5+k^4+2k-2}$$

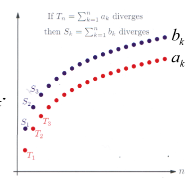
Comparison Test

★ Given two series of positive terms $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $a_k \leq b_k$ for all $k \in \mathbb{Z}^+$, then:

- If $\sum_{k=1}^{\infty} b_k$ is convergent to a limit S, then $\sum_{k=1}^{\infty} a_k$ is also convergent to a limit T where $T \leq S$.



- If $\sum_{k=1}^{\infty} a_k$ is divergent, then so is $\sum_{k=1}^{\infty} b_k$.



Ex 3:

Establish whether or not the series $\sum_{k=1}^{\infty} \frac{1}{2^k+3}$ converges.

Limit Comparison Test

Given two series of positive terms $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$,

where $\lim_{k\to\infty} \frac{a_k}{b_k} = l > 0$, then:

- If one series converges so does the other.
- If one series diverges so does the other.

Limit Comparison Test

Show that the series $\sum_{k=1}^{\infty} \frac{1}{2^k-1}$ is convergent.

1. Show that the following converge using the Comparison Test:

(a) (i) $\sum_{k=1}^{\infty} \frac{1}{k^2+5}$ (ii) $\sum_{k=1}^{\infty} \frac{1}{k(k+3)}$

(b) (i) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+1}$ (ii) $\sum_{k=1}^{\infty} \frac{k}{k^4+2k+4}$

(c) (i) $\sum_{k=3}^{\infty} \frac{k-2}{k^3+3}$ (ii) $\sum_{k=2}^{\infty} \frac{\sqrt{k-1}}{k^2+3k+5}$

(d) (i) $\sum_{k=1}^{\infty} \frac{1}{k^k}$ (ii) $\sum_{k=1}^{\infty} \frac{1}{k!}$

(e) (i) $\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k+5^k}$ (ii) $\sum_{k=1}^{\infty} \frac{2^k}{e^{2k}}$

(f) (i) $\sum_{k=1}^{\infty} \frac{\cos^2 k}{k^2}$ (ii) $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2+2}$

2. Show that the following diverge using the Comparison Test or Divergence Test:

(a) (i) $\sum_{k=4}^{\infty} \frac{1}{k-3}$ (ii) $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k-1}}$

(b) (i) $\sum_{k=1}^{\infty} \frac{k^2}{2k^2+7k+4}$ (ii) $\sum_{k=1}^{\infty} \frac{3k+2}{2k+1}$

(c) (i) $\sum_{k=2}^{\infty} \frac{k+1}{\sqrt{k^4-2}}$ (ii) $\sum_{k=2}^{\infty} \frac{k^2+3}{k^3-2}$

(d) (i) $\sum_{k=1}^{\infty} \frac{4^{k+2}}{3^k}$ (ii) $\sum_{k=1}^{\infty} \frac{(\pi+k)^{k+1}}{3^k}$

(e) (i) $\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k}$ (ii) $\sum_{k=1}^{\infty} \frac{\sqrt{k^4+1}}{k}$

(f) (i) $\sum_{k=1}^{\infty} \left(\frac{x+k}{k}\right)^k$ (ii) $\sum_{k=1}^{\infty} \left(\frac{k}{x+k}\right)^k$

3. Determine whether the following converge or diverge using the Limit Comparison Test.

(a) (i) $\sum_{k=2}^{\infty} \frac{4}{k^3-2}$ (ii) $\sum_{k=1}^{\infty} \frac{1}{k+3}$

(b) (i) $\sum_{k=1}^{\infty} \frac{1}{5^k-4}$ (ii) $\sum_{k=1}^{\infty} \frac{2^k}{3^k-1}$

(c) (i) $\sum_{k=1}^{\infty} \frac{1}{2k+\sqrt{k}+3}$ (ii) $\sum_{k=2}^{\infty} \frac{1}{4k^2-k-1}$