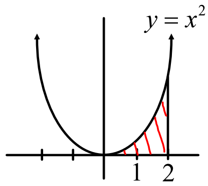
First Fundamental Theorem of Calculus

$$F(x)=\int_a^x f(t)dt \qquad \text{where} \quad F'(x)=f(x)$$

- States the relationship between the integral of a function and its derivative.
- The derivative and integral of a function are opposite operations (undo each other).

Second Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b)-F(a) \qquad \text{where} \quad F'(x)=f(x)$$



$$\begin{aligned} Area &= \int_0^2 x^2 dx \\ &= \left[\frac{1}{3}x^3\right]_0^2 \\ &= \frac{1}{3}\left(2^3-0^3\right) \\ &= \frac{8}{3} \end{aligned}$$

Definition Review

Convergent = Approaches a specific, non-infinite value.

Convergent Series: 8,4,2,1,... $\sum_1^\infty 5\left(\frac{1}{3}\right)^n$

Convergent Function: $\lim_{x\to\infty}\frac{2}{3x+1}$ $\lim_{n\to\infty}\frac{n+1}{n}$

Divergent = Does not approach a specific value.

Divergent Series: 1,3,9,27,... $\sum_1^\infty 4(3)^n$

Divergent Function: $\lim_{x\to\infty}x^2$ $\lim_{n\to\infty}\frac{n^3}{n+12}$

Classifying Integrals

Indefinite Integral $\int x^2 dx$

Definite Integral $\int_0^2 x^2 dx$

Improper Integral $\int_0^\infty x^2 dx$

Classifying Integrals (Cont.)

Indefinite Integral $\int x^2 dx$ $\frac{1}{3}x^3+c$

What is it?

"Undoes" the derivative to give you the family of functions whose derivative is shown in the integral.

Classifying Integrals (Cont.)

Definite Integral $\int_0^2 x^2 dx$ $\frac{8}{3}$

What is it?

Area between the function and the x-axis on the interval defined by the lower and upper boundaries of the integral (0 and 2 in this example).

Classifying Integrals (Cont.)

Improper Integral $\int_0^\infty x^2 dx$?

What is it?

Area between the function and the x-axis from a defined lower boundary to an infinite upper boundary.

Although these have an "infinite" upper boundary, many do have "finite" areas.

We are essentially saying that if the area gets small *fast enough* it can be approximated by a finite value.

Evaluating Improper Integrals

Since indefinite integrals do not have an upper boundary, we need a different way to evaluate them. This is where limits come into play.

Ex1: $\int_1^\infty \frac{1}{x^2} dx =$

$=$

$=$

$=$

$=$

Evaluating Improper Integrals

Ex2: $\int_0^\infty e^{-3x} dx$

Important Prior Limits: $\lim_{x\to\infty} e^{-x} = 0$, $\lim_{x\to\infty} x^{-p} = 0, p > 0$

Improper Integrals with Powers of x (IMPORTANT!!!)

Ex3: $\int_1^\infty x^p dx = \lim_{b\to\infty} \int_1^b x^p dx$

Improper Integrals with Powers of x (IMPORTANT!!!)

Summary:

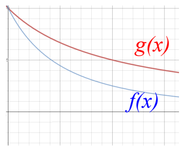
$\int_1^\infty x^p dx$ *convergent* only when $p < -1$

or

$\int_1^\infty \frac{1}{x^p} dx$ *convergent* only when $p > 1$

Comparison Test for Improper Integrals

If $0 \le f(x) \le g(x)$ for all $x > a$, then:



If $\int_a^\infty g(x) dx$ is convergent, then

If $\int_a^\infty f(x) dx$ is divergent, then

Comparison Test (Cont.)

$0 \le f(x) \le g(x)$ ⟵ Note: When choosing a usable *g(x)*, it is very easy to find a *g(x)* that is always larger than *f(x)* for all $x > a$.

BUT, you must make sure it is larger than *f(x)* AND still converges. This is the challenge!

Usually, we have fractions or expressions that can be written as fractions. To ensure one fraction is larger than another, you can:

1. Make the numerator larger
2. Make the denominator smaller
3. Both (1) and (2).

Comparison Test (Cont.)

Ex4: Show $\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$ is convergent.

Comparison Test Extension

If $\int_a^\infty |f(x)|\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is convergent.

This is extremely useful for functions that oscillate above and below the *x*-axis such as sin and cos functions.

Comparison Test (Cont.)

Ex5: Show $\int_1^\infty \frac{\cos x}{1+x^3}\,dx$ converges.

$\int_1^\infty \frac{1}{x^3}\,dx$ converges ($\int_1^\infty \frac{1}{x^p}\,dx, p > 1$)

Thus, by the comparison test $\int_1^\infty \left|\frac{\cos x}{1+x^3}\right| dx$ converges.

By extension, $\int_1^\infty \frac{\cos x}{1+x^3}\,dx$ also converges.

1. Test the following for convergence. For those that do converge, give the value of the improper integral.

(a) $\int_0^\infty \frac{1}{(1+x)^2}\,\mathrm{d}x$ (b) $\int_0^\infty e^{-\frac{x}{4}}\,\mathrm{d}x$

1. Test the following for convergence. For those that do converge, give the value of the improper integral.

(c) $\int_0^\infty \frac{1}{\sqrt{1+x}}\,\mathrm{d}x$ (d) $\int_0^\infty x e^{-x^2}\,\mathrm{d}x$

2. Find which of the following converge using the Comparison Test for improper integrals:

(a) $\int_2^\infty \frac{1}{\sqrt{x}-1}\,dx$ (b) $\int_1^\infty e^{-x^2}\,dx$

2. Find which of the following converge using the Comparison Test for improper integrals:

(c) $\int_1^\infty \frac{2x}{x^3+5}\,dx$ (d) $\int_1^\infty \frac{\cos\frac{x}{2}}{x^2+4x}\,dx$

2. Find which of the following converge using the Comparison Test for improper integrals:

(e) $\int_3^\infty \frac{x+2}{x^2-3x+1}\,dx$ (f) $\int_1^\infty \frac{\sin 2x}{x^4+x^2+2}\,dx$

3. Find for what values of p the following improper integrals converge:

(a) $\int_0^\infty e^{px}\,dx$ (b) $\int_1^\infty \frac{\ln x}{x^p}\,dx$

4. Evaluate $\int_0^\infty e^{-x}\,dx$. *[3 marks]*

5. Evaluate $\int_0^\infty \frac{1}{x^2+1}\,dx$. *[3 marks]*

6. (a) Show that *[5 marks]*

$x^4 + 6x^2 + 14 = (x^2 + 3)^2 + 5$ for all $x \in \mathbb{R}$

(b) Hence show that

$\int_1^{\infty} \frac{x^2 + 3}{x^4 + 6x^2 + 14}\,dx$ converges

7. (a) Show that *[5 marks]*

$$\int_a^4 \frac{1}{\sqrt{x}}\,dx = 4 - 2\sqrt{a}$$

for $0 < a < 4$.

(b) Hence explain why $\int_0^4 \frac{1}{\sqrt{x}}\,dx$ converges and find its value.

8. (a) Show that $\frac{3}{x^2 + x - 2} = \frac{1}{x - 1} - \frac{1}{x + 2}$. *[6 marks]*

(b) Hence find the exact value of:

$\int_3^{\infty} \frac{3}{x^2 + x - 2}\,dx$

9. Evaluate the improper integral: $\int_{1/\pi}^{\infty} \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx$ *[6 marks]*

10. Show that $\int_0^{\infty} \frac{\sin(\lambda x)}{e^x}\,dx = \frac{\lambda}{1 + \lambda^2}$. *[10 marks]*

11. Show by induction that $\int_0^{\infty} x^n e^{-x}\,dx = n!$ *[11 marks]*

12. Does the improper integral $\int_1^\infty \frac{\sin x}{x}\,dx$ converge? *[9 marks]*