Homework

1G p.39-40 #1-3,5,6,8,9

Applications of Continuity and Differentiability

Continuity $\lim\limits_{x \to x_o} f(x) = f(x_0)$

Limit as the function approaches a value is equal to the function evaluated at that point.

Differentiability
1.) Continuous on the given interval
2.) No "sharp" points
3.) No vertical tangents

Differentiable $\xrightarrow{\text{Always}}$ Continuous

Continuous $\xrightarrow{\text{Could Be}}$ Differentiable

Intermediate Value Theorem (IVT)

For a function *f(x)* that is:

« Continuous on the interval $[a,b]$

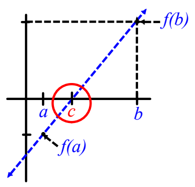
« Has values at the ends of the interval *f(a)* and *f(b)*

Then there exists some number *c* such that:

« $a < c < b$

« $f(a) < f(c) < f(b)$

Extension: If $f(a) < 0$ and $f(b) > 0$, then there must exist an $f(c) = 0$.



Intermediate Value Theorem (IVT)

If
- f is a continuous function on the interval $[a,b]$
- $f(a) < 0$ and $f(b) > 0$

Then
- there must exist a value $c \in]a,b[$ where $f(c) = 0$.

If a function has at least one negative and one positive value, then there must exist a third point between the two original points where the graph crosses the *x*-axis

(aka: *x*-int, root, zero, solution).

Mean Value Theorem (MVT)

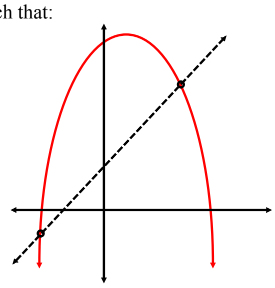
For a function *f(x)* that is:

1.) Continuous on the interval $[a,b]$

2.) Differentiable on the interval $]a,b[$

There must exist a point $c \in]a,b[$ such that:

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$



Mean Value Theorem (MVT)

If
- f is a continuous function on the interval $[a,b]$
- f is a differentiable function on the interval $]a,b[$

Then
- there must exist a value $c \in]a,b[$ where

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

If a function is continuous and differentiable, then there must be a point on the interval where the instantaneous rate of change is equal to the average rate of change over the same interval.

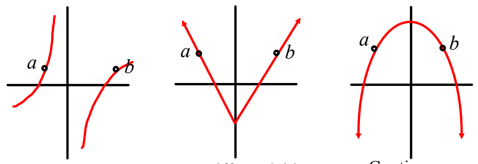
Rolle's Theorem (Extension of MVT)

For a function *f(x)* that is:

1.) Continuous on the interval $[a,b]$
(Function must exist at ends of interval)

2.) Differentiable on the interval $]a,b[$
(Not necessarily at a and b)

If $f(a) = f(b)$, then there must exist some $c \in]a,b[$ such that $f'(c)=0$.



Not Continuous — Not Differentiable — Continuous and Differentiable

Rolle's Theorem

If

- f is a continuous function on the interval $[a,b]$
- f is a differentiable function on the interval $]a,b[$
- $f(a) = f(b)$

Then

- there must exist a value $c \in]a,b[$ where $f'(c) = 0$.

If a function is continuous and differentiable and the function values at the ends of the interval are equal, then there must be a point on the interval where a max or min exists.

Rolle's Theorem (Example 1)

Prove $f(x) = x^3 + 3x^2 + 6x + 1$ has exactly one real root.

1.) VERIFY EXISTENCE OF AT LEAST 1 REAL ROOT USING IVT.

$f(0) = 1$
$f(-1) = -3$
THUS, BY THE IVT THERE EXIST AT LEAST 1 REAL ROOT ON $[-1,0]$.

2.) PROVE NO OTHER ROOTS EXIST USING ROLLE'S THEOREM.

LET a BE THE FIRST REAL ROOT ON $[-1,0]$.
ASSUME b IS A SECOND REAL ROOT. SO,
$f(a) = f(b) = 0$. THUS, THERE MUST EXIST A POINT AT $x=c$ SUCH THAT $f'(c)=0$.

$f'(x) = 3x^2 + 6x + 6$

$3x^2 + 6x + 6 = 0$

$x = \dfrac{-6 \pm \sqrt{36 - 4(3)(6)}}{6}$

$\boxed{x = -1 \pm i}$

SINCE $f'(c) \neq 0$ AT ANY REAL VALUE, THEN OUR ORIGINAL ASSUMPTION $f(b)=0$ IS FALSE.

$\therefore$ $f(x)$ HAS ONLY 1 REAL ROOT.

MVT (Example 2)

Determine all numbers c which satisfy the conclusions of the Mean Value Theorem for the following function:

$f(x) = x^3 + 2x^2 - x$ on the interval $[-1,2]$

$f(-1) = 2$
$f(2) = 14$

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$3x^2 + 4x - 1 = \dfrac{14 - 2}{2 - (-1)}$

$3x^2 + 4x - 1 = 4$

$3x^2 + 4x - 5 = 0$

$x = \dfrac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6}$

$x = \dfrac{-4 \pm \sqrt{76}}{6}$

$x = \dfrac{-2 \pm \sqrt{19}}{3} \Rightarrow x \approx 0.786, \cancel{-2.120}$

$[-1, 2]$

MVT (Example 3)

If $f(x)$ is such that $f(2) = -4$ and $f'(x) \geq 2$ for all $x \in]2,7[$, find the smallest possible value for $f(7)$.

$f(2) = -4$
$f(7) = ?$

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f'(c) = \dfrac{f(7) - (-4)}{7 - 2}$

$f'(c) = \dfrac{f(7) + 4}{5}$

$f(7) = 5 \cdot f'(c) - 4$

$f'(x) \geq 2$

$f(7) = 5 \cdot 2 - 4$

$f(7) = 6$ SMALLEST POSSIBLE VALUE

Mean Value Theorem (Example 4)

A car driving along the motorway and travelling below the speed limit of 70 mph passes a police officer at 12:00. At 12:20 the same car passes another police officer 24 miles further down the road along the motorway, again travelling less than 70 mph. The driver is pulled over by the second policeman and given a speeding ticket.

Use the Mean Value Theorem to show how the police knew the driver had exceeded the posted speed limit.

$f(0) = 0$
$f\left(\frac{1}{3}\right) = 24$

MVT

$\underbrace{f'(c)}_{\text{RATE OF CHANGE (SPEED) AT POINT } c \in]0,\frac{1}{3}[} = \dfrac{24 - 0}{\frac{1}{3} - 0} = \underbrace{72 \text{ mph}}_{\text{AVG. SPEED}}$

SINCE THE DISTANCE TRAVELLED IS CONTINUOUS, THE MVT STATES THAT, AT SOME POINT IN THE 20 MIN INTERVAL, YOU MUST HAVE BEEN TRAVELLING AT YOUR AVERAGE SPEED OF 72 mph.