Vectors (Revisited)

Given the points $P(2,-7,-1)$ and $R(-1,4,5)$:

a.) Find $|\overrightarrow{PR}|$. $\overrightarrow{PR} = \begin{pmatrix} -3 \\ 11 \\ 6 \end{pmatrix} = \sqrt{166}$

b.) Find the vectors with magnitude 10 collinear to $\overrightarrow{PR}$.

$v = \pm \frac{10}{\sqrt{166}} \begin{pmatrix} -3 \\ 11 \\ 6 \end{pmatrix}$

c.) Find the equation of a line (in all 3 forms) containing the points $P(2,-7,-1)$ and $R(-1,4,5)$.

Vector Eqn

$r = \begin{pmatrix} 2 \\ -7 \\ -1 \end{pmatrix} + a \begin{pmatrix} -3 \\ 11 \\ 6 \end{pmatrix}$

Parametric Eqns

$x = 2 - 3a$
$y = -7 + 11a$
$z = -1 + 6a$

Cartesian Eqns

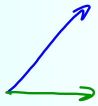
$\frac{x-2}{-3} = \frac{y+7}{11} = \frac{z+1}{6}$

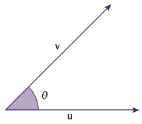
Dot Product/Scalar Product

$$u \cdot v = |u| \cdot |v| \cdot \cos\theta$$

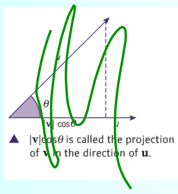
u dot v =

- u and v are non-zero vectors
- θ is the angle between u and v.





To measure the angle between two vectors, the vectors must have the same initial point.



$|v|\cos\theta$ is called the projection of v in the direction of u.

Important: The dot product returns a scalar value, *NOT* a vector.

Dot Product (Alternate Forms)

$$\cos\theta = \frac{u \cdot v}{|u| \cdot |v|} \qquad \theta = \cos^{-1}\left(\frac{u \cdot v}{|u| \cdot |v|}\right)$$

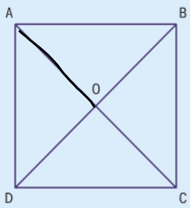
Properties of the Dot Product $u \cdot v = |u| \cdot |v| \cdot \cos\theta$

→ Here are some important consequences of the geometric definition of the scalar product.

- The scalar product of two vectors is always a number.
- The definition of scalar product does not depend on the dimensions of the space.
- $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal**.
- $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}|\,|\mathbf{v}|$ if $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
- $\mathbf{u} \cdot \mathbf{v} > 0$ when θ is acute and $\mathbf{u} \cdot \mathbf{v} < 0$ when θ is obtuse.
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(\lambda\,\mathbf{u}) \cdot \mathbf{v} = \lambda(\mathbf{u} \cdot \mathbf{v})$

Example #2

Consider the unit square ABCD. Let O be the point where the diagonals of the square meet. Find $\overrightarrow{OA} \cdot \overrightarrow{AB}$



$\sqrt{2}$

$135°$

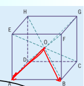
$u \cdot v = |u||v| \cos\theta$

$\left(\frac{\sqrt{2}}{2}\right)(1)\left(-\frac{\sqrt{2}}{2}\right)$

$\boxed{-\frac{1}{2}}$

Example #3

Consider the unit cube ABCDEFGH. Let O be the point where its four diagonals meet. Find $\overrightarrow{OA} \cdot \overrightarrow{OB}$



O $\angle C$, $a = \frac{\sqrt{3}}{2}$, $b = \frac{\sqrt{3}}{2}$, A, B, $c = 1$

$c^2 = a^2 + b^2 - 2ab\cos\angle C$

$1^2 = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$ $\cos\angle O$

$1 = \frac{6}{4} - \frac{6}{4}\cos\angle O$

$-\frac{1}{2} = -\frac{3}{2}\cos\angle O$

$\frac{1}{3} = \cos\angle O$

$\overrightarrow{OA} \cdot \overrightarrow{OB} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{3} = \boxed{\frac{1}{4}}$

$|a + bi| = \sqrt{a^2 + b^2}$

$\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| =$

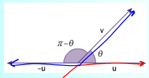
$z = 6 + 2i$

$\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

Homework

11I p.586-7 #1-8

2 Use the diagram to show that given two non-zero vectors **u** and **v**, $\mathbf{u}\cdot(-\mathbf{v}) = -(\mathbf{u}\cdot\mathbf{v}) = (-\mathbf{u})\cdot\mathbf{v}$.



$u\cdot(-v) = |u||-v|\cos(\pi-\theta)$

$-(u\cdot v) = |u||v|\cos\theta$

$(-u)\cdot v = |-u||v|\cos(\pi-\theta)$

$|u| = |-u|$

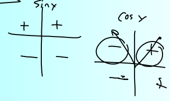
$|v| = |-v|$

$u\cdot(-v) = |u||v|\cos(\pi-\theta)$

$-(u\cdot v) = |u||v|(-\cos\theta)$

$(-u)\cdot v = |u||v|\cos(\pi-\theta)$

$= \cos$



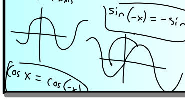
EVEN vs. ODD FXNS

$f(x) = f(-x)$

Ref. sym y-axis

$f(-x) = -f(x)$

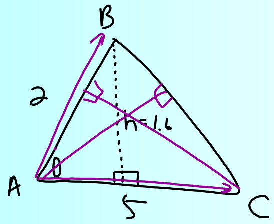
$\sin(-x) = -\sin x$



$\cos x = \cos(-x)$

5 A triangle ABC has area 4.

Given that AB = 2 and AC = 5, find the possible values of $\overrightarrow{AB}\cdot\overrightarrow{AC}$.



$\frac{\sin\angle A}{a} = \frac{\sin\angle B}{b}$

$\cos\theta = \frac{1.2}{2} = 0.6$

$(3\text{-}4\text{-}5)\times 4$

$(12\text{-}16\text{-}20)\div 10$

$1.2\text{-}1.6\text{-}2.0$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|\cdot\cos\theta$

$= 2\cdot 5\cdot\cos\theta$

$= 2\cdot 5\cdot 0.6$

$= \boxed{6}$

Exercise 11I

7 Use the result in question **6** to show that given two vectors **u** and **v**, $\mathbf{u}\cdot\mathbf{v} = 0$ if $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}-\mathbf{v}|$

$u\cdot v = 0$ $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$u\cdot u = |u|^2$

$u+v = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix}$

$u-v = \begin{pmatrix} u_1-v_1 \\ u_2-v_2 \end{pmatrix}$

$|u+v| = \sqrt{(u_1+v_1)^2+(u_2+v_2)^2}$

$|u-v| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$

$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$

$(u_1+v_1)^2+(u_2+u_2)^2 = (u_1-v_1)^2+(u_2-v_2)^2$

$\cancel{u_1^2} + 2u_1v_1 + \cancel{v_1^2} + \cancel{u_2^2} + 2u_2v_2 + \cancel{v_2^2} =$

$\cancel{u_1^2} - 2u_1v_1 + \cancel{v_1^2} + \cancel{u_2^2} - 2u_2v_2 + \cancel{v_2^2}$

$2u_1v_1 + 2u_2v_2 = -2u_1v_1 - 2u_2v_2$

$2(u_1v_1+u_2v_2) = -2(u_1v_1+u_2v_2)$

$4(u_1v_1+u_2v_2) = 0$

$u_1v_1+u_2v_2 = 0$

$\boxed{u\cdot v = 0}$

Dot Product/Scalar Product

Given two vectors in a plane:

~~$u = u_1\mathrm{i} + u_2\mathrm{j}$~~

~~$v = v_1\mathrm{i} + v_2\mathrm{j}$~~

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$u\cdot v = u_1v_1 + u_2v_2$

Given two vectors in 3D:

~~$u = u_1\mathrm{i} + u_2\mathrm{j} + u_3\mathrm{k}$~~

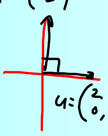
~~$v = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k}$~~

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$u\cdot v = u_1v_1 + u_2v_2 + u_3v_3$

$v = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$



$u = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$u\cdot v = 2\cdot 0 + 0\cdot 2$

Example 1

Find $\mathbf{u}\cdot\mathbf{v}$ when:

a $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$u\cdot v = u_1v_1 + u_2v_2$

$= 2(-1) + 3(-2)$

$= -2 - 6$

$= \boxed{-8}$

Example #1

Find the scalar product of the vectors **u** and **v** given that $|\mathbf{u}| = 2$, $|\mathbf{v}| = 3$, and the angle between **u** and **v** is 60°.

dot

$u \cdot v = 2 \cdot 3 \cdot \frac{1}{2} = \boxed{3}$

11J p.589 #1-4

Example 2

Find $\mathbf{u} \cdot \mathbf{v}$ when:

b $\mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$

$u \cdot v = -1(-4) + 2(-1) - 3(2)$

$u \cdot v = 4 - 2 - 6$

$u \cdot v = -4$

Example 3

Given the points A(–1, 0, 2), B(0, 1, –2) and C(1, 1, 1), find $\overrightarrow{AB} \cdot \overrightarrow{BC}$.

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

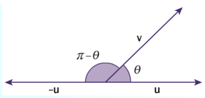
$\overrightarrow{AB} \cdot \overrightarrow{BC} = 1 + 0 - 12 = \boxed{-11}$

Exercise 11I

1 Find the scalar product of the vectors **u** and **v** given that $|\mathbf{u}| = 1.5$, $|\mathbf{v}| = 4$ and the angle between **u** and **v** is 30°.

Exercise 11I

2 Use the diagram to show that given two non-zero vectors **u** and **v**, $\mathbf{u} \cdot (-\mathbf{v}) = -(\mathbf{u} \cdot \mathbf{v}) = (-\mathbf{u}) \cdot \mathbf{v}$.



Exercise 11I

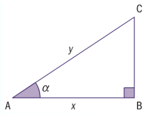
3 Consider an equilateral triangle ABC. Find $\overrightarrow{AB} \cdot \overrightarrow{BC} + \overrightarrow{BC} \cdot \overrightarrow{AC}$.

Exercise 11I

4 In right-angled triangle ABC, $\hat{A} = \alpha$, $\hat{B} = 90°$, AB $= x$ and AC $= y$.

Find, in terms of x, y and α

a $\overrightarrow{AB} \cdot \overrightarrow{AC}$ **b** $\overrightarrow{CA} \cdot \overrightarrow{CB}$ **c** $\overrightarrow{AC} \cdot \overrightarrow{CB}$



Exercise 11I

5 A triangle ABC has area 4.

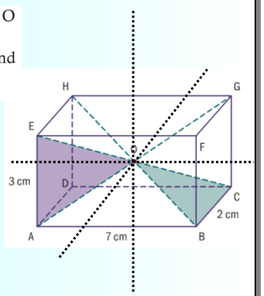
Given that AB = 2 and AC = 5, find the possible values of $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

Exercise 11I

8 Consider the cuboid ABCDEFGH with centre O shown here.

Using the information given on the diagram, find

a $\overrightarrow{OB} \cdot \overrightarrow{OC}$ **b** $\overrightarrow{AO} \cdot \overrightarrow{OE}$



Exercise 11J

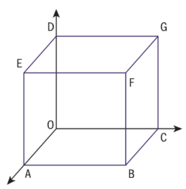
1 Find $\mathbf{u} \cdot \mathbf{v}$ when

a $\mathbf{u} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 12 \\ -6 \end{pmatrix}$ **b** $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

11J p.589 #1-4

2 Given the points A(–1, 3, –2), B(–1, 1, 2) and C(1, –1, 1), find $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and $\overrightarrow{AC} \cdot \overrightarrow{BC}$.

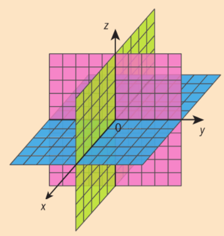
3 The unit cube OABCDEFG in the diagram has its faces parallel to the coordinate planes.



a Write down the coordinates of its vertices.

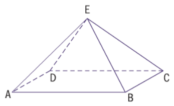
b Hence, find $\overrightarrow{OF} \cdot \overrightarrow{OG}$ and $\overrightarrow{AF} \cdot \overrightarrow{BG}$.

The coordinate planes are mutually perpendicular and intersect at the origin O.



EXAM-STYLE QUESTION

4 Consider a square-based pyramid ABCDE such that the x-axis contains B and D, the y-axis contains A and C and the positive part of the z-axis contains E.



Given that the area of the base is 4 square units and the volume of the pyramid is $\frac{8}{3}$ cubic units, find

a the coordinates of its vertices

b $|\overrightarrow{EA}|$ and $\overrightarrow{EA} \cdot \overrightarrow{EB}$

c the size of angle $A\hat{E}B$.