First Fundamental Theorem of Calculus

\[ F(x) = \int_a^b f(t) \, dt \quad \text{where} \quad F'(x) = f(x) \]

- States the relationship between the integral of a function and its derivative.
- The derivative and integral of a function are opposite operations (undo each other).

Second Fundamental Theorem of Calculus

\[ \int_a^b f(x) \, dx = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x) \]

**Definition Review**

**Convergent** = Approaches a specific, non-infinite value.
- Convergent Series: \( 8, 4, 2, 1, \ldots \)
- Convergent Function: \( \lim_{n \to \infty} \frac{2}{3x+1} \)
- \( \lim_{n \to \infty} \frac{n+1}{n} \)

**Divergent** = Does not approach a specific value.
- Divergent Series: \( 1, 3, 9, 27, \ldots \)
- Divergent Function: \( \lim_{x \to \infty} x^2 \)
- \( \lim_{n \to \infty} \frac{n}{n+12} \)

**Classifying Integrals**

**Indefinite Integral** \( \int x^2 \, dx \)

**Definite Integral** \( \int_0^2 x^2 \, dx \)

**Improper Integral** \( \int_0^\infty x^2 \, dx \)

**Classifying Integrals (Cont.)**

**Indefinite Integral** \( \int x^2 \, dx \cdot \frac{1}{3} x^3 + c \)

**Definite Integral** \( \int_0^2 x^2 \, dx \cdot \frac{8}{3} \)

- Area between the function and the x-axis on the interval defined by the lower and upper boundaries of the integral (0 and 2 in this example).
Classifying Integrals (Cont.)

Improper Integral \( \int_0^\infty x^3 \, dx \)

What is it?

Area between the function and the x-axis from a defined lower boundary to an infinite upper boundary.

Although these have an "infinite" upper boundary, many do have "finite" areas.

We are essentially saying that if the area gets small fast enough, it can be approximated by a finite value.

Evaluating Improper Integrals

Ex1: \( \int_1^\infty \frac{1}{x^2} \, dx \)

Upper limit becomes the variable \( b \), and we evaluate the limit as \( b \) approaches infinity.

Evaluating Improper Integrals

Ex2: Important Prior Limits

Improper Integrals with Powers of \( x \) (IMPORTANT!!)

Ex3: \( \int_1^\infty x^p \, dx \)

Summary:

\[ \int_1^\infty x^p \, dx \text{ convergent only when } p < -1 \]

or

\[ \int_1^\infty \frac{1}{x^p} \, dx \text{ convergent only when } p > 1 \]

Comparison Test for Improper Integrals

If \( 0 \leq f(x) \leq g(x) \) for all \( x > a \), then:

If \( \int_a^\infty g(x) \, dx \) is convergent, then \( \int_a^\infty f(x) \, dx \) is convergent.

If \( \int_a^\infty f(x) \, dx \) is divergent, then \( \int_a^\infty g(x) \, dx \) is divergent.
Comparison Test (Cont.)

\[ 0 \leq f(x) \leq g(x) \]  

Note: When choosing a usable \( g(x) \), it is very easy to find a \( g(x) \) that is always larger than \( f(x) \) for all \( x > a \).

BUT, you must make sure it is larger than \( f(x) \) AND still converges. This is the challenge!

Usually, we have fractions or expressions that can be written as fractions. To ensure one fraction is larger than another, you can:

1. Make the numerator larger
2. Make the denominator smaller
3. Both (1) and (2).

Comparison Test Extension

If \( \int_a^\infty f(x) \, dx \) is convergent, then \( \int_a^\infty f(x) \, dx \) is convergent.

This is extremely useful for functions that oscillate above and below the \( x \)-axis such as sin and cos functions.

Comparison Test (Cont.)

Ex4: Show \( \int_a^\infty \frac{\sqrt{x}}{1 + x^2} \, dx \) is convergent.

Comparison Test (Cont.)

Ex5: Show \( \int_a^\infty \frac{\cos x}{1 + x^2} \, dx \) converges.

Thus, by the comparison test \( \int_a^\infty \frac{\cos x}{1 + x^2} \, dx \) converges.

By extension, \( \int_a^\infty \frac{\cos x}{1 + x^2} \, dx \) also converges.

1. Test the following for convergence. For those that do converge, give the value of the improper integral.

(a) \( \int_a^\infty \frac{1}{1 + x^2} \, dx \)  
(b) \( \int_a^\infty e^{-x^2} \, dx \)
2. Find which of the following converge using the Comparison Test for improper integrals:

(a) \[ \int_{1}^{\infty} \frac{1}{\sqrt{x} - 1} \, dx \]  
(b) \[ \int_{1}^{\infty} e^{-x^2} \, dx \]  
(c) \[ \int_{1}^{\infty} \frac{2x}{x^2 + 3} \, dx \]  
(d) \[ \int_{1}^{\infty} \frac{\cos \frac{1}{x}}{x^2 + 4x} \, dx \]  

3. Find for what values of \( p \) the following improper integrals converge:

(a) \[ \int_{0}^{\infty} e^{-x^p} \, dx \]  
(b) \[ \int_{1}^{\infty} \frac{\ln x}{x^p} \, dx \]  

4. Evaluate \[ \int_{0}^{\infty} e^{-x} \, dx \]. [3 marks]
6. (a) Show that
\[ x^4 + 6x^2 + 14 = (x^2 + 3)^2 + 5 \] for all \( x \in \mathbb{R} \)

(b) Hence show that\[ \int_{-\infty}^{\infty} \frac{x^2 + 3}{x^4 + 6x^2 + 14} \, dx \] converges

7. (a) Show that \[ \int_{\sqrt{a}}^{1} \frac{1}{\sqrt{x}} \, dx = 4 - 2\sqrt{a} \] for \( 0 < a < 4 \)

(b) Hence explain why \[ \int_{\sqrt{a}}^{1} \frac{1}{\sqrt{x}} \, dx \] converges and find its value.

8. (a) Show that \( \frac{3}{x^2 + x - 2} = \frac{1}{x - 1} - \frac{1}{x + 2} \). \[ \text{[6 marks]} \]

(b) Hence find the exact value of: \[ \int_{1}^{2} \frac{3}{x^2 + x - 2} \, dx \]

9. Evaluate the improper integral \[ \int_{1}^{\infty} \frac{1}{x^2 - \sin \left( \frac{1}{x} \right)} \, dx \] \[ \text{[6 marks]} \]

10. Show that \[ \int_{0}^{\infty} \frac{\sin(\lambda x)}{e^x} \, dx = \frac{\lambda}{1 + \lambda^2} \]. \[ \text{[10 marks]} \]

11. Show by induction that \[ \int_{0}^{1} x^n e^{-x} \, dx = n! \] \[ \text{[11 marks]} \]
12. Does the improper integral \( \int_{1}^{\infty} \frac{\sin x}{x} \, dx \) converge? [9 marks]